Linear and Numerical Linear Algebra Dictionary

Matrices, Factorizations, and Numerical Methods

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UNIVERSITY OF TRENTO

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1. Basic Notation

Purpose and Scope

This is not a set of lecture notes. It is a compact dictionary of definitions and results used in the accompanying courses. If you want to study any of these topics, you should follow a course or consult a textbook devoted to the subject. This dictionary is intended for readers who already know, or have previously studied, the material and need a concise reference whenever a reminder is useful.

The following $A \in \mathbb{K}^{m\times n}$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For square plates $A \in \mathbb{K}^{n\times n}$:

$$A^T = \text{transpose}, \qquad A^* = \overline{A}^T = \text{conjugate transpose}.$$

In the real field $A^* = A^T$.

$$\operatorname{spec}(A) = \{\lambda \in \mathbb{C} : \det(A - \lambda I) = 0\}, \qquad \rho(A) = \max_{\lambda \in \operatorname{spec}(A)} |\lambda|.$$



2. Summary Table

Type	Defining condition	Eigenvalues	Key properties
Diagonal	$a_{ij} = 0$ for $i \neq j$	diagonal elements	diagonal inverse if not singular
Scale	$A = \alpha I$	all equal to α	switch with every matrix
Triangular	$a_{ij} = 0$ above or below the diagonal	diagonal elements	$\det A = \prod_i a_{ii}$
Real symmetric	$A^T = A$	real	diagonally orthogonally
Real antisymmetrical	$A^T = -A$	pure imaginary or 0	diagonal nothing
Hermitian	$A^* = A$	real	unitarily diagonalizable
Antihermitiana	$A^* = -A$	pure imaginary or 0	iA Hermitian
Orthogonal	$A^T A = I$	$\lvert\lambda\rvert = 1$	$A^{-1} = A^T$, $\det A = \pm 1$
Unitarian	$A^* A = I$	$\lvert\lambda\rvert = 1$	$A^{-1} = A^*$
Normal	$A^* A = A A^*$	complexes in general	unitarily diagonalizable
SPD	$A = A^T$, $x^T A x > 0$	positive	Cholesky, invertible
SPSD	$A = A^T$, $x^T A x \geq 0$	nonnegative	Gram, square root
negative definite	$A = A^T$, $x^T A x < 0$	negative	$-A$ SPD
Indefinite	square shape of variable sign	positive and negative	not semidefinite
Idempotent	$A^2 = A$	$0, 1$	projector, $\operatorname{rank} A = \operatorname{tr} A$
Involution	$A^2 = I$	± 1	$A^{-1} = A$
Nilpotent	$A^k = 0$	only 0	$\det A = 0$
Permutation	one unit per line/column	roots of unity	orthogonal
Householder	$H = I - 2vv^T/(v^T v)$	± 1	symmetric and orthogonal
Toeplitz	$a_{ij} = t_{i-j}$	—	constant diagonals
Hankel	$a_{ij} = h_{i+j}$	—	constant antidiagonal

Type	**Defining condition**	**Eigenvalues**	**Key properties**
Circulating	rows obtained for cyclic shift	DFT of coefficients	diagonalized by DFT
Vandermonde	$V_{ij} = x_i^{j-1}$	—	known instrument norm matrix for polynomials
Companion	associated with a polynomial	roots of polynomial	
Stocha	elements ≥ 0, row/column sums $= 1$	contains $\lambda = 1$	$\rho(A) = 1$
Gram Matrix	$G = B^*B$	nonnegative	positive semidefinite Hermitian



3. Inner Products

3.1 Real Inner Product

A product to scale up $\mathbb{R}^n$ is a function

$$\langle x, y \rangle$$

that satisfies:

- bilinearity;
- symmetry;
- positivity:

$$\langle x, x \rangle \geq 0;$$

- defined:

$$\langle x, x \rangle = 0 \iff x = 0.$$

Product euclide inner:

$$\langle x, y \rangle = x^T y.$$

3.2 Complex Inner Product

Up $\mathbb{C}^n$:

$$\langle x, y \rangle = x^* y.$$

Property:

$$\langle x, y \rangle = \overline{\langle y, x \rangle}.$$

3.3 Weighted Inner Product

$$M = M^* \succ 0,$$

then

$$\langle x, y \rangle_M = x^* M y.$$

The associated norm is:

$$\|x\|_M = \sqrt{x^* M x}.$$



4. Vector Norms

A vector rule meets:

$$\|x\| \geq 0,$$
$$\|x\| = 0 \iff x = 0,$$
$$\|\alpha x\| = |\alpha| \, \|x\|,$$
$$\|x+y\| \leq \|x\| + \|y\|.$$

4.1 1-Norm

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|.$$

4.2 2-Norm

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2} = \sqrt{x^* x}.$$

4.3 Infinity Norm

$$\|x\|_\infty = \max_i |x_i|.$$

4.4 p-Norm

For

$$1 \leq p < \infty :$$
$$\|x\|_p = \left(\sum_i |x_i|^p \right)^{1/p}.$$

4.5 Cauchy–Schwarz Inequality

For norms induced by scalar product:

$$|\langle x, y \rangle| \leq \|x\| \, \|y\|.$$

4.6 Equivalence of Norms in Finite Dimensions

Up $\mathbb{K}^n$ all norms are equivalent.

In particular:

$$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\,\|x\|_\infty,$$
$$\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\,\|x\|_2.$$



5. Matrix Norms

A matrix norm meets the properties of a norm on the matrix space.

A matrix rule is submultiplicative if:

$$\|AB\| \leq \|A\| \, \|B\|.$$

5.1 Induced Matrix Norm

Date a vector rule:

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \|Ax\|.$$

Induced norms are automatically compatible with the vector norm:

$$\|Ax\| \leq \|A\| \, \|x\|.$$

5.2 Matrix 1-Norm

$$\|A\|_1 = \max_j \sum_i |a_{ij}|.$$

It is the maximum of the sum of the absolute values along the columns.

5.3 Matrix Infinity Norm

$$\|A\|_\infty = \max_i \sum_j |a_{ij}|.$$

It is the maximum of the sum of the absolute values along the rows.

5.4 Matrix 2-Norm

$$\|A\|_2 = \sigma_{\max}(A) = \sqrt{\lambda_{\max}(A^* A)}.$$

A is normal:

$$\|A\|_2 = \rho(A).$$

A is symmetric/Hermitian:

$$\|A\|_2 = \max_i |\lambda_i|.$$

5.5 Frobenius Norm

$$\|A\|_F = \sqrt{\sum_{i,j} |a_{ij}|^2}.$$

Equivalently:

$$\|A\|_F = \sqrt{\mathrm{tr}(A^* A)} = \sqrt{\sum_i \sigma_i^2}.$$

Property:

$$\|A\|_2 \leq \|A\|_F \leq \sqrt{\mathrm{rank}(A)}\, \|A\|_2.$$

Frobenius is not a norm induced by a vector norm, but is submultiplicative:

$$\|AB\|_F \leq \|A\|_F \|B\|_F.$$

In addition:

$$\|AB\|_F \leq \|A\|_2 \|B\|_F,$$
$$\|AB\|_F \leq \|A\|_F \|B\|_2.$$



6. Compatibility between Norms

A matrix norm $\|\cdot\|_M$ is compatible with a vector rule $\|\cdot\|_v$ if:

$$\|Ax\|_v \leq \|A\|_M \|x\|_v.$$

Any induced matrix rule is compatible with the vector norm from which it is induced. For norm norms:

$$\|Ax\|_1 \leq \|A\|_1 \|x\|_1,$$
$$\|Ax\|_2 \leq \|A\|_2 \|x\|_2,$$
$$\|Ax\|_\infty \leq \|A\|_\infty \|x\|_\infty.$$



7. Norms and Inner Products

Each inner product induces a norm:

$$\|x\| = \sqrt{\langle x,x\rangle}.$$

Not every norm is induced by a scalar product.
A norm is induced by a scalar product if and only if it meets the identity of the parallelogram:

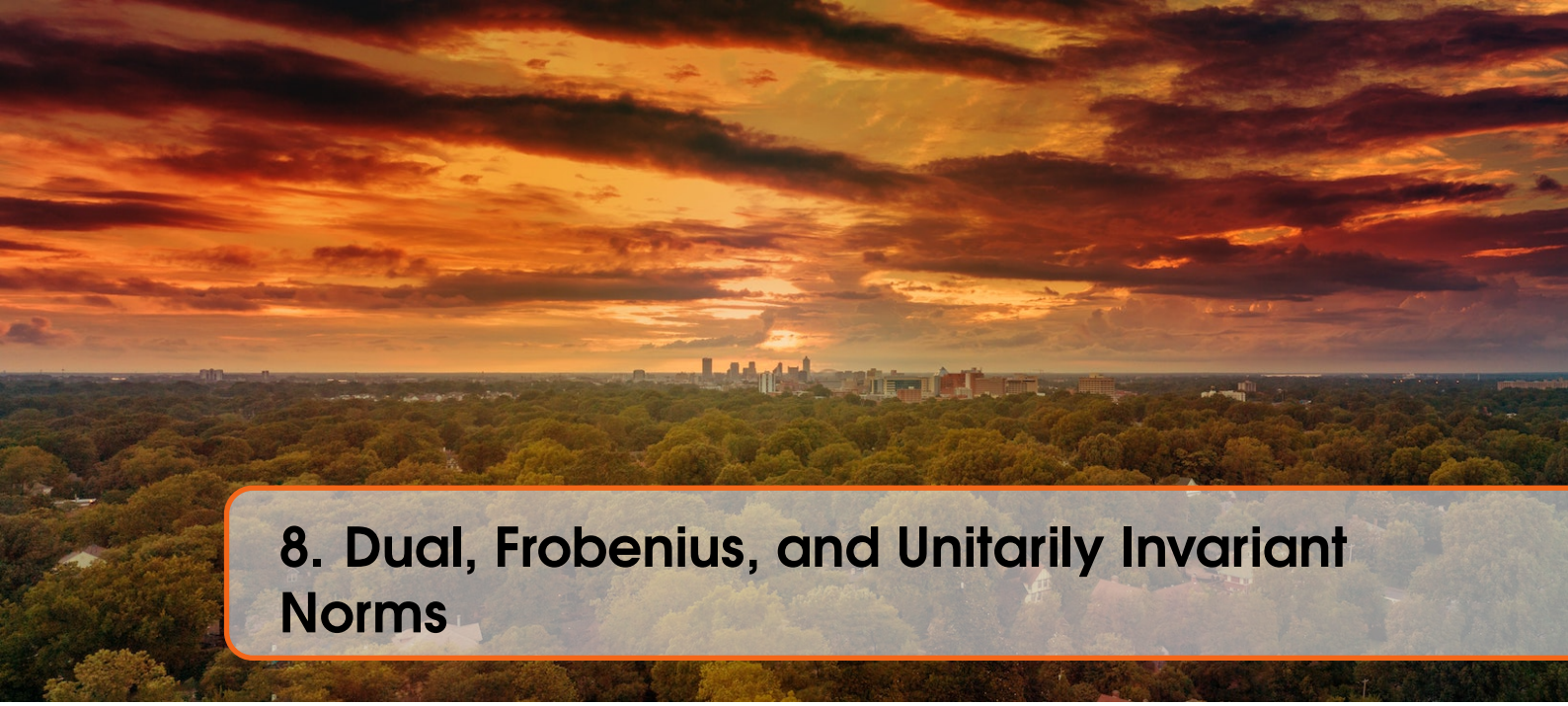
$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

For the Euclidean rule:

$$\|x\|_2 = \sqrt{x^* x}.$$

For a positive SPD/Hermitian matrix M:

$$\|x\|_M = \sqrt{x^* M x}.$$



8. Dual, Frobenius, and Unitarily Invariant Norms

The dual rule is ’

$$\|y\|_* = \max_{\|x\|\le 1} |y^* x|.$$

$1/p + 1/q = 1$:

$$(\|\cdot\|_p)^* = \|\cdot\|_q, \qquad |x^* y| \le \|x\|_p \|y\|_q.$$

Frobenius inner product

$$\langle A, B\rangle_F = \operatorname{tr}(A^* B), \qquad \|A\|_F = \sqrt{\operatorname{tr}(A^* A)}.$$

For U, V Units:

$$\|UAV\|_2 = \|A\|_2, \qquad \|UAV\|_F = \|A\|_F.$$

Schatten rules:

$$\|A\|_{S_p} = \left(\sum_i \sigma_i^p\right)^{1/p}.$$

Nuclear norms:

$$\|A\|_* = \sum_i \sigma_i.$$



9. Fundamental Subspaces and Rank

Both $A \in \mathbb{K}^{m\times n}$ and $\operatorname{rank}(A) = r$. The four basic subspaces are

$$\operatorname{range}(A) \subseteq \mathbb{K}^m, \qquad \ker(A) \subseteq \mathbb{K}^n,$$

$$\operatorname{range}(A^*) \subseteq \mathbb{K}^n, \qquad \ker(A^*) \subseteq \mathbb{K}^m.$$

The size is

$$\dim \operatorname{range}(A) = \dim \operatorname{range}(A^*) = r, \quad \dim \ker(A) = n - r, \quad \dim \ker(A^*) = m - r.$$

Theorem rank–nulled':

$$\operatorname{rank}(A) + \dim \ker(A) = n.$$

Basic orthogonal relations:

$$\operatorname{range}(A)^\perp = \ker(A^*), \qquad \operatorname{range}(A^*)^\perp = \ker(A),$$

$$\mathbb{K}^m = \operatorname{range}(A) \oplus \ker(A^*), \qquad \mathbb{K}^n = \operatorname{range}(A^*) \oplus \ker(A).$$



10. Rank and Invertibility: Equivalent Conditions

For $A \in \mathbb{K}^{n\times n}$ are equivalent:

A invertible,
$\operatorname{rank}(A) = n$,
$\det(A) \neq 0$,
$0 \notin \operatorname{spec}(A)$,
$\ker(A) = \{0\}$,
$Ax = 0 \Rightarrow x = 0$,
the columns of A are linearly independent,
the rows of A are linearly independent.



11. Rectangular Matrices

11.1 Full-Rank Matrix

For $A \in \mathbb{K}^{m\times n}$:

$$\operatorname{rank}(A) = \min(m,n).$$

$m \geq n$ and $\operatorname{rank}(A) = n$, A has full rank for columns.
$m \leq n$ and $\operatorname{rank}(A) = m$, A has full rank for rows.

11.2 Matrix with Orthonormal Columns

$$Q^*Q = I.$$

Q is rectangular, in general:

$$QQ^* \neq I.$$

QQ^* is the orthogonal projector on $\operatorname{range}(Q)$.

11.3 Matrix with Orthonormal Rows

$$QQ^* = I.$$

In this case Q^*Q He's an orthogonal projector.



12. Basic Properties of the Determinant, Trace, and Spectrum

For $A, B \in \mathbb{K}^{n\times n}$:

$$\det(AB) = \det(A)\det(B),$$
$$\det(A^T) = \det(A), \qquad \det(A^*) = \overline{\det(A)},$$
$$\det(A^{-1}) = \frac{1}{\det A},$$
$$\operatorname{tr}(AB) = \operatorname{tr}(BA),$$
$$\operatorname{tr}(A) = \sum_{i=1}^{n} \lambda_i,$$
$$\det(A) = \prod_{i=1}^{n} \lambda_i,$$

where the eigenvalues are counted with a multiplicity of algebraic.

$B = S^{-1}AS$, then A and B are similar and have:

- same characteristic polynomial;
- same eigenvalues;
- itself decisive;
- same trace;
- same rank.



13. Eigenvalues, Spectrum, and Spectral Radius

$$Av = \lambda v, \qquad ve0, \qquad p_A(\lambda) = \det(\lambda I - A).$$

For each eigenvalue:

$$1 \leq m_g(\lambda) \leq m_a(\lambda).$$

Useful transformations:

$$\operatorname{spec}(A^{-1}) = \{\lambda^{-1} : \lambda \in \operatorname{spec}(A)\}, \quad \operatorname{spec}(A + \alpha I) = \operatorname{spec}(A) + \alpha,$$

$$\operatorname{spec}(\alpha A) = \alpha \operatorname{spec}(A), \qquad \operatorname{spec}(A^k) = \{\lambda^k : \lambda \in \operatorname{spec}(A)\}.$$

For each polynomial p:

$$\operatorname{spec}(p(A)) = p(\operatorname{spec}(A)).$$

The spectral radius is

$$\rho(A) = \max_{\lambda \in \operatorname{spec}(A)} |\lambda|.$$

For each induced matrix rule:

$$\rho(A) \leq \|A\|.$$

Gelfand formula:

$$\rho(A) = \lim_{k \to \infty} \|A^k\|^{1/k}.$$

In finite dimensions:

$$A^k \to 0 \iff \rho(A) < 1.$$

13.1 Gershgorin's Theorem

$$\operatorname{spec}(A) \subseteq \bigcup_i \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{jei} |a_{ij}| \right\}.$$



14. Spectral Theorem and Rayleigh Quotient

For $A = A^*$:

$$A = U\Lambda U^*, \qquad \Lambda \in \mathbb{R}^{n\times n}.$$

In the symmetric real case:

$$A = Q\Lambda Q^T.$$

Rayleigh quotient is

$$R_A(x) = \frac{x^*Ax}{x^*x}.$$

For $A = A^*$:

$$\lambda_{\min} \le R_A(x) \le \lambda_{\max},$$

$$\lambda_{\min} = \min_{x\neq 0} R_A(x), \qquad \lambda_{\max} = \max_{x\neq 0} R_A(x).$$

Ordering $\lambda_1 \le \cdots \le \lambda_n$ Courant–Fischer provides

$$\lambda_k = \min_{\dim S = k} \max_{0\neq x\in S} R_A(x) = \max_{\dim S = n-k+1} \min_{0\neq x\in S} R_A(x).$$



15. Diagonalizability and Canonical Forms

15.1 Diagonalizable Matrix

A is diagonalized if there is an invertible matrix V such that

$$A = V\Lambda V^{-1},$$

with Λ diagonal.

Equival to the existence of a base constituted by motorists.

Sufficient conditions:

- A he has n distinct eigenvalues;
- A is normal on $\mathbb{C}$;
- A is real symmetric.

15.2 Defective Matrix

A non- diagonalized matrix is called defective. For at least one eigenvalue:

$$\dim\ker(A-\lambda I) < \text{algebraic multiplicity of } \lambda.$$

15.3 Jordan Form

Each complex matrix is similar to a block matrix:

$$A = VJV^{-1},$$

where J consists of Jordan blocks

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}.$$

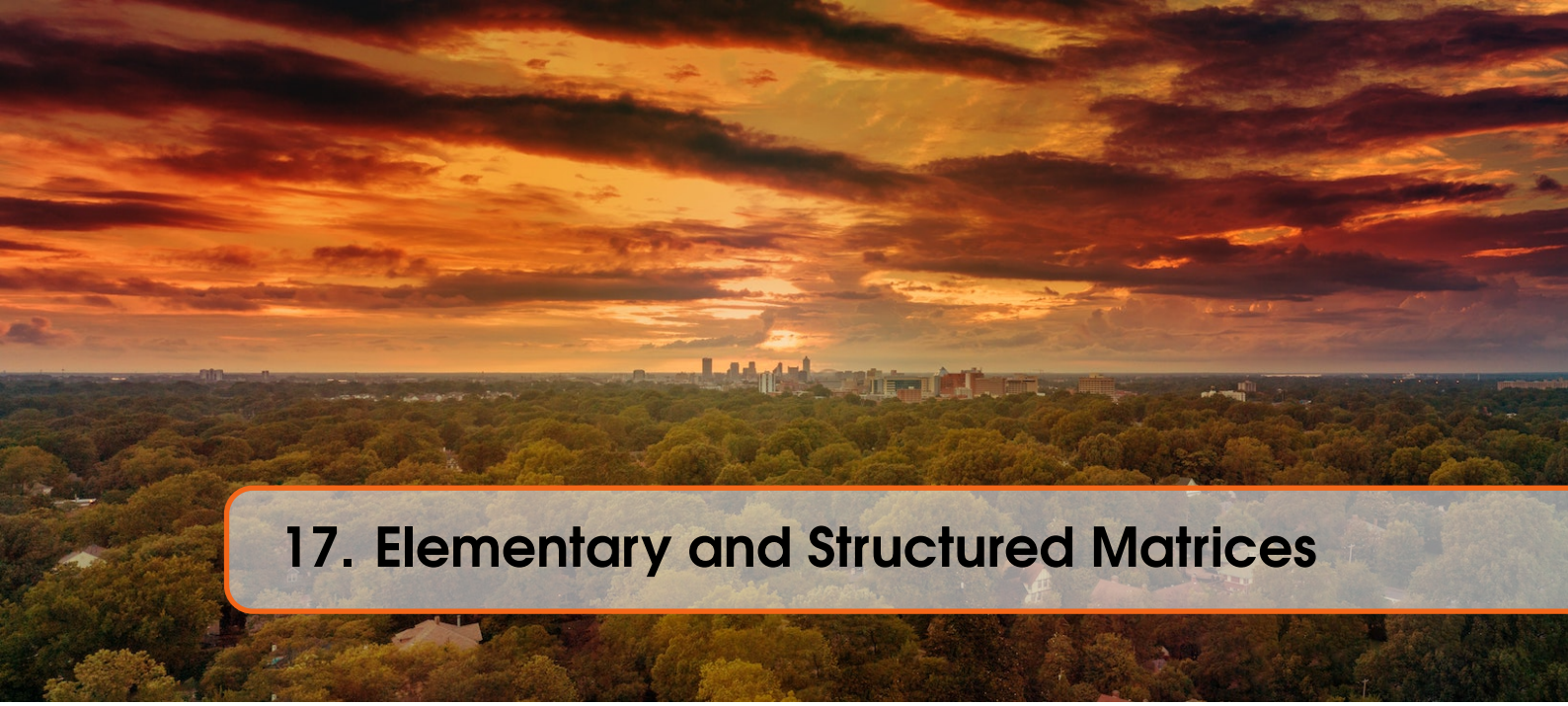


16. Schur Decomposition

For each $A \in \mathbb{C}^{n\times n}$:

$$A = QTQ^*,$$

with Q unitary and T superior triangular. The diagonal elements of T are the eigenvalues of A. In the real case there is a form of real Schur with diagonal blocks 1×1 and 2×2. A is normal, T is diagonal. Schur's form is numerically stable; Jordan's form is mainly theoretical.



17. Elementary and Structured Matrices

17.1 Zero Matrix

$$A = 0.$$

Property:

- $\operatorname{rank}(A) = 0$.
- $\operatorname{spec}(A) = \{0\}$.
- A is nilpotent, symmetric/Hermitian, normal and semidefinite positive and negative.

17.2 Identity Matrix

$$I_n = \operatorname{diag}(1, \ldots, 1).$$

Property:

- $I_n A = A I_n = A$.
- $I_n^{-1} = I_n$.
- $\operatorname{spec}(I_n) = \{1\}$.
- It is diagonal, symmetric, Hermitian, orthogonal, unitary, normal, SPD.

17.3 Diagonal Matrix

$$D = \operatorname{diag}(d_1, \ldots, d_n).$$

Property:

- $\operatorname{spec}(D) = \{d_1, \ldots, d_n\}$.
- $\det D = \prod_i d_i$.
- $\operatorname{rank} D =$ number of diagonal elements not null.
- D is invertible if and only if $d_i \neq 0$ for each i.
- In this case

$$D^{-1} = \operatorname{diag}(d_1^{-1}, \ldots, d_n^{-1}).$$

17.4 Scalar Matrix

$$A = \alpha I.$$

Property:

- switched with each square matrix of the same size;
- $\operatorname{spec}(A) = \{\alpha\}$;

- $\det A = \alpha^n$;
- is invertible if and only if $\alpha \neq 0$.

17.5 Upper and Lower Triangular Matrices

Upper triangular:

$$a_{ij} = 0 \qquad i > j.$$

Lower triangular:

$$a_{ij} = 0 \qquad i < j.$$

Property:

- $\det A = \prod_i a_{ii}$.
- The eigenvalues are the diagonal elements.
- It is invertible if and only if $a_{ii} \neq 0$ for each i.
- The inverse of a not singular triangular is triangular of the same type.
- The triangular matrix product of the same type is triangular.

17.6 Strictly Triangular Matrix

Triangular with diagonal nothing. Property:

- is nilpotent;
- $A^n = 0$ for a matrix $n \times n$;
- All eigenvalues are null.

17.7 Bidiagonal Matrix

It has elements not null only on the main diagonal and on a diagonal immediately adjacent.

17.8 Tridiagonal Matrix

$$a_{ij} = 0 \qquad |i - j| > 1.$$

Property:

- tridiagonal linear systems can be solved in $O(n)$ operations;
- if symmetric, it is completely described by the main diagonal and an adjacent diagonal.

17.9 Band Matrix

Ha.

$$a_{ij} = 0 \qquad \text{for} \qquad i - j > p \quad \text{or} \quad j - i > q.$$

$p$and$q$ are the lower and upper band semi-ampies.

17.10 Hessenberg Matrix

Upper Hessenberg:

$$a_{ij} = 0 \qquad i > j + 1.$$

Lower Hessenberg:

$$a_{ij} = 0 \qquad j > i + 1.$$

Each square matrix can be reduced by unit/orthogonal similarities in the shape of Hessenberg.

17.11 Block Matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

The usual matrix operations apply to blocks when the dimensions are compatible.

A_{11} is invertible, Schur's complement is

$$S = A_{22} - A_{21}A_{11}^{-1}A_{12}.$$



18. Symmetry, Adjoints, and Normality

18.1 Real Symmetric Matrix

$$A^T = A.$$

Property:

- all eigenvalues are real;
- autovectors related to separate autovalves are orthogonal;
- exists an orthogonal matrix Q such that

$$A = Q\Lambda Q^T;$$

- is always diagonalized;
- $\|A\|_2 = \rho(A) = \max_i |\lambda_i|$.

18.2 Real Skew-Symmetric Matrix

$$A^T = -A.$$

Property:

- $a_{ii} = 0$;
- eigenvalues are null or pure imaginary;
- if n is odd, $\det A = 0$;
- $x^T Ax = 0$ for each $x \in \mathbb{R}^n$.

18.3 Hermitian Matrix

$$A^* = A.$$

Property:

- real eigenvalues;
- autovectors associated with distinct eigenvalues are orthogonal;
- exist U such that

$$A = U\Lambda U^*, \qquad \Lambda \in \mathbb{R}^{n\times n}.$$

18.4 Skew-Hermitian Matrix

$$A^* = -A.$$

Property:

- eigenvalues are pure or null imaginary;
- iA is Hermitian;
- e^A is unitary.

18.5 Normal Matrix

$$A^*A = AA^*.$$

Fundamental equivalences:

- A is normal if and only if it is diagonally diagonally;
- exist U such that
$$A = U\Lambda U^*.$$

They're normal.

- hermitiane matrices;
- antihermitiane;
- unit;
- real symmetries;
- real orthogonals.



19. Orthogonal and Unitary Matrices

19.1 Orthogonal Matrix

$$Q^T Q = QQ^T = I.$$

Property:

- $Q^{-1} = Q^T$;
- preserves scalable product, angles and Euclidean norm:

$$\|Qx\|_2 = \|x\|_2;$$

- $|\lambda_i| = 1$;
- $\det Q = \pm 1$;
- $\text{cond}_2(Q) = 1$.

19.2 Unitary Matrix

$$U^* U = UU^* = I.$$

Property:

- $U^{-1} = U^*$;
- preserves hermitian product;
- $\|Ux\|_2 = \|x\|_2$;
- $|\lambda_i| = 1$;
- $|\det U| = 1$.

19.3 Rotation Matrix

In $\mathbb{R}^2$:

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Property:

$$R^T R = I, \qquad \det R = 1.$$

19.4 Householder Reflection

For $v \neq 0$:

$$H = I - 2\frac{vv^T}{v^T v}.$$

Property:

- $H^T = H$;
- $H^T H = I$;
- $H^{-1} = H$;
- $H^2 = I$;
- $\det H = -1$;
- eigenvalues: -1 in the direction of v, $+1$ in the orthogonal subspace.

19.5 Givens Rotation

It acts not banally on two coordinates:

$$G = \begin{pmatrix} 1 & & & & \\ & c & & -s & \\ & & 1 & & \\ & s & & c & \\ & & & & 1 \end{pmatrix}, \qquad c^2 + s^2 = 1.$$

It is orthogonal and is used to selectively cancel elements of a matrix.



20. Definiteness

20.1 Positive Definite Matrix

For a symmetric real matrix:

$$A \succ 0 \quad \Longleftrightarrow \quad x^T Ax > 0 \quad \forall x \neq 0.$$

Equivalent properties:

- all eigenvalues are positive;
- exists the factorization of Cholesky
$$A = LL^T$$
with L lower triangular and positive diagonal;
- all major minors"leading"are positive (Sylvester crime);
- $A = B^T B$ with B invertable;
- $A^{-1} \succ 0$.

In addition:

$$\det A > 0.$$

20.2 Positive Semidefinite Matrix

$$A \succeq 0 \quad \Longleftrightarrow \quad x^T Ax \geq 0 \quad \forall x.$$

Property:

- non-negative eigenvalues;
- exist B such that
$$A = B^T B;$$
- admits a positive symmetric square root:
$$A^{1/2} A^{1/2} = A.$$

20.3 Negative Definite Matrix

$$A \prec 0 \quad \Longleftrightarrow \quad x^T Ax < 0 \quad \forall x \neq 0.$$

Equival to

$$-A \succ 0.$$

eigenvalues are all negative.

20.4 Negative Semidefinite Matrix

$$A \preceq 0 \quad \Longleftrightarrow \quad x^T A x \leq 0.$$

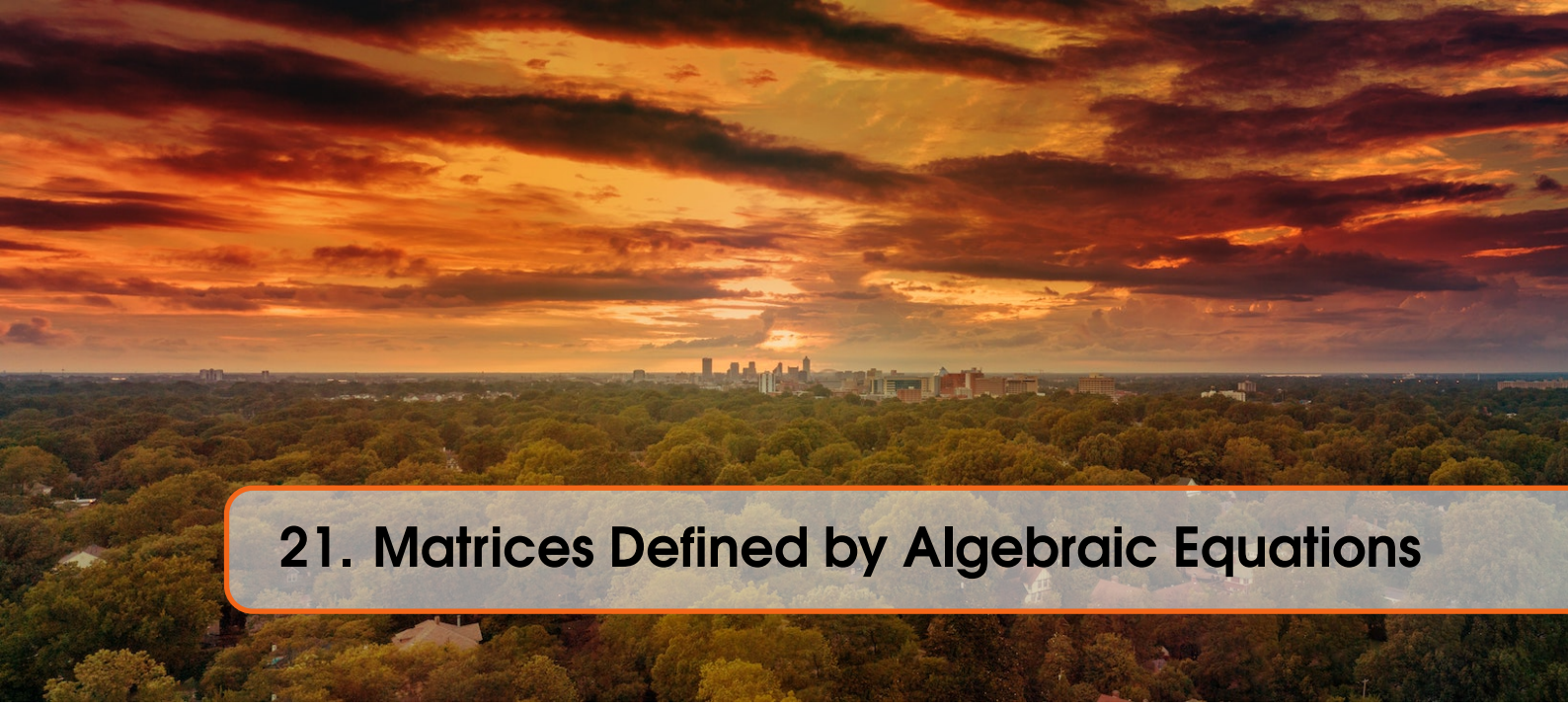
eigenvalues are not positive.

20.5 Indefinite Matrix

There are $x, y \neq 0$ such as

$$x^T A x > 0, \qquad y^T A y < 0.$$

For a symmetric matrix this is equivalent to the presence of at least one positive eigenvalue and at least one negative.



21. Matrices Defined by Algebraic Equations

21.1 Idempotent Matrix

$$P^2 = P.$$

Property:

- eigenvalues in $\{0,1\}$;
- is diagonalized;
- $$\operatorname{rank} P = \operatorname{tr} P;$$
- represents a projection.

21.2 Orthogonal Projector

$$P^2 = P, \qquad P^T = P.$$

Property:

- P project orthogonally on $\operatorname{range}(P)$;
- $$\mathbb{R}^n = \operatorname{range}(P) \oplus \ker(P);$$
- $\|P\|_2 = 1$ if $P \neq 0$.

A has linearly independent columns:

$$P = A(A^T A)^{-1} A^T$$

is the orthogonal projector on $\operatorname{range}(A)$.

21.3 Oblique Projector

$$P^2 = P, \qquad P^T \neq P.$$

The projection takes place along a subspace not necessarily orthogonal to the image subspace.

21.4 Involutory Matrix

$$A^2 = I.$$

Property:

- $A^{-1} = A$;
- eigenvalues in $\{-1, 1\}$;
- is diagonalized on different characteristic fields from 2.

21.5 Nilpotent Matrix

Existing $k \geq 1$ such that

$$A^k = 0.$$

Property:

- only eigenvalue: 0;
- $\det A = 0$;
- $\mathrm{tr} A = 0$;
- $I - A$ is always invertible;
- if $A^k = 0$:

$$(I - A)^{-1} = I + A + \cdots + A^{k-1}.$$

21.6 Unipotent Matrix

$$A = I + N,$$

with N nilpotent. All eigenvalues of A are equal to 1.



22. Permutation and Sparse Matrices

22.1 Permutation Matrix

It is obtained by switching rows or columns I. Property:

- contains exactly one 1 in each row and column;
- $$P^{-1} = P^T;$$
- is orthogonal;
- $\det P = \pm 1$;
- multiply to the left permutate rows, to the right permutate columns.

22.2 Sparse Matrix

A matrix is called scattered when the number of nonzero elements is much lower than mn. The notion is computational, not algebraic.

22.3 Diagonally Dominant Matrix

Dominance by rows:

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|.$$

Directly dominant:

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|.$$

A strictly diagonally dominant matrix is not singular.



23. Classical Special Matrices

23.1 Gram Matrix

For carriers $v_1, \ldots, v_n$:

$$G_{ij} = \langle v_i, v_j \rangle.$$

Equivalently:

$$G = V^* V.$$

Property:

- G is Hermitian;
- $G \succeq 0$;
- $G \succ 0$ if and only if the carriers are linearly independent;
-

$$\operatorname{rank}(G) = \operatorname{rank}(V).$$

23.2 Vandermonde Matrix

$$V = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}.$$

determinant:

$$\det V = \prod_{1 \le i < j \le n} (x_j - x_i).$$

It is invertible if and only if x_i They're distinct.

23.3 Toeplitz Matrix

$$a_{ij} = t_{i-j}.$$

The elements are constant along each diagonal.

Shape:

$$T = \begin{pmatrix} t_0 & t_{-1} & t_{-2} & \cdots \\ t_1 & t_0 & t_{-1} & \cdots \\ t_2 & t_1 & t_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

23.4 Hankel Matrix

$$a_{ij} = h_{i+j}.$$

The elements are constant along the antidiagonals.

23.5 Circulant Matrix

$$C = \begin{pmatrix} c_0 & c_{n-1} & \cdots & c_1 \\ c_1 & c_0 & \cdots & c_2 \\ \vdots & & \ddots & \vdots \\ c_{n-1} & c_{n-2} & \cdots & c_0 \end{pmatrix}.$$

Property:

- is diagonalized by Fourier's discreet matrix;
- eigenvalues are the DFT of the carrier $(c_0, \ldots, c_{n-1})$;
- all circular matrices of the same size switch between them.

23.6 Companion Matrix

For monic polynomial

$$p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0,$$

a common convention

$$C = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}.$$

The polynomial characteristic of C It is p.

23.7 Cauchy Matrix

$$C_{ij} = \frac{1}{x_i + y_j},$$

when $x_i + y_j \neq 0$. It has explicit formulas for determinant and inverse and is often very numerically ill-conditioned.

23.8 Hilbert Matrix

$$H_{ij} = \frac{1}{i + j - 1}.$$

Property:

- symmetric;
- positive definition;
- strongly badly conditioned for n growing.



24. Nonnegative and Stochastic Matrices

24.1 Nonnegative Matrix

$$A \geq 0$$

means

$$a_{ij} \geq 0 \quad \forall i, j.$$

Inequality is a component.

24.2 Positive Matrix

$$A > 0$$

means

$$a_{ij} > 0 \quad \forall i, j.$$

Not to be confused with"definita positiva".

24.3 Row-Stochastic Matrix

$$a_{ij} \geq 0, \qquad \sum_j a_{ij} = 1.$$

Property:

- $$A\mathbf{1} = \mathbf{1};$$
- 1 is eigenvalue;
- $\rho(A) = 1$.

24.4 Column-Stochastic Matrix

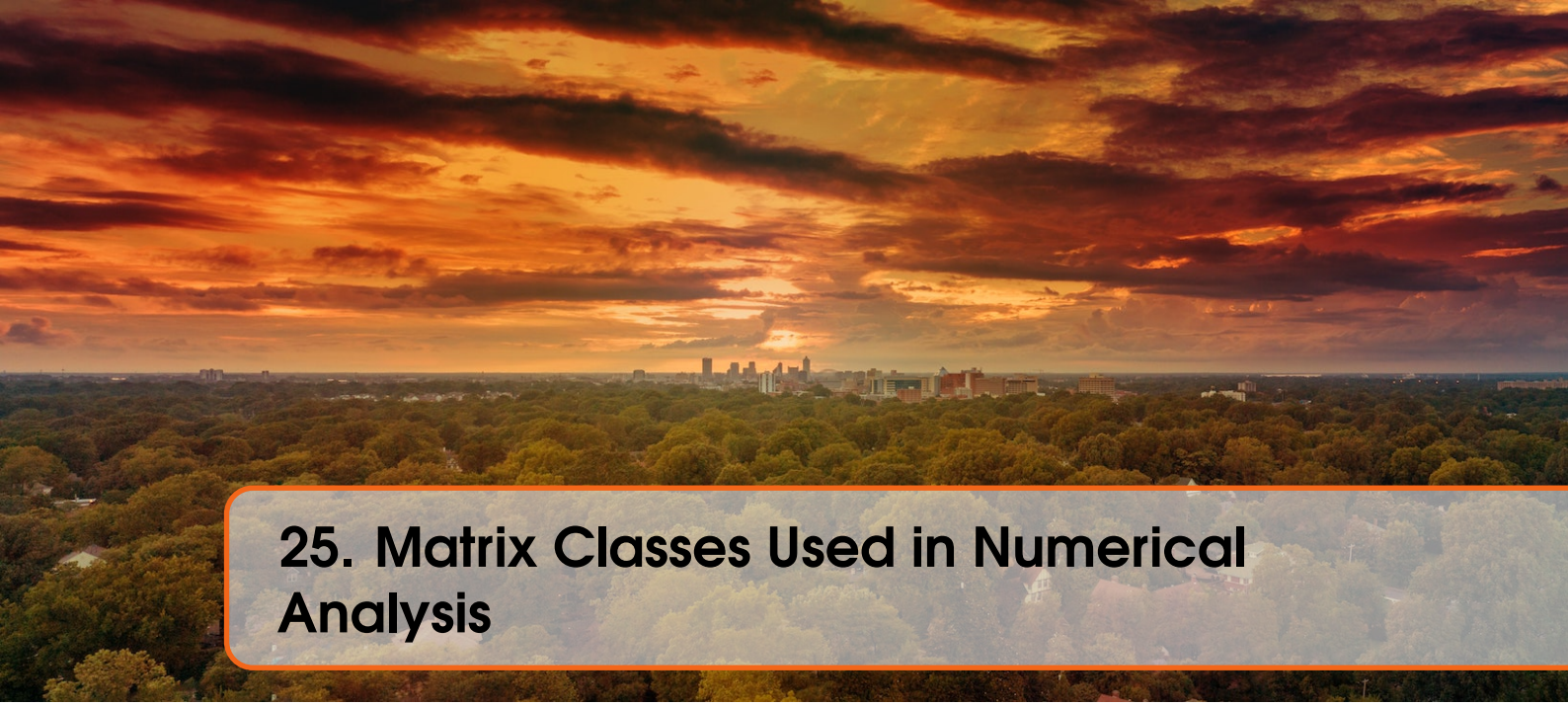
$$a_{ij} \geq 0, \qquad \sum_i a_{ij} = 1.$$

Then

$$\mathbf{1}^T A = \mathbf{1}^T.$$

24.5 Doubly Stochastic Matrix

It is stochastic for both rows and columns. Every permutation matrix is bistocastic.



25. Matrix Classes Used in Numerical Analysis

25.1 Z-Matrix

A real matrix A is a Z- Matrix if

$$a_{ij} \leq 0 \qquad i \neq j.$$

25.2 M-Matrix

One M-matrix can be represented as

$$A = sI - B, \qquad B \geq 0, \qquad s \geq \rho(B).$$

$$s > \rho(B),$$

is a M- Not unusual matrix.

For one M- not singular matrix:

- $A^{-1} \geq 0$;
- eigenvalues have a positive real part;
- A is a Z- Matrix.

25.3 P-Matrix

A square real matrix is a P-Matrix if all its major minors are positive.

Property:

- every SPD matrix is a P-matrix;
- I'll give you the P- matrices have a central role in the problems of linear complementarity.

25.4 Monotone Matrix

A not singular real matrix A is called monotonous if

$$A^{-1} \geq 0.$$

25.5 Irreducible Matrix

A square matrix is reducible if there is a permutation matrix P such that

$$P^T AP = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}.$$

Otherwise it is irreducible.



26. Important Relationships between Matrix Classes

$$\boxed{A \text{ SPD} \implies A \text{ symmetric} \implies A \text{ normal} \implies A \text{ diagonalizable}}$$

$$\boxed{A \text{ Hermitian} \implies A \text{ normal} \implies A \text{ unitarily diagonalizable}}$$

$$\boxed{A \text{ unitary} \implies A \text{ normal}}$$

$$\boxed{A \text{ orthogonal} \implies A \text{ normal}}$$

$$\boxed{A^2 = A \implies \operatorname{spec}(A) \subseteq \{0,1\}}$$

$$\boxed{A^2 = I \implies \operatorname{spec}(A) \subseteq \{-1,1\}}$$

$$\boxed{A^k = 0 \implies \operatorname{spec}(A) = \{0\}}$$

$$\boxed{A = B^*B \implies A \succeq 0}$$

$$\boxed{A = B^*B,\ \operatorname{rank}(B) = n \implies A \succ 0}$$



27. Appendix: Terminology Not to Confuse

- **matrix positiva**:

$$a_{ij} > 0.$$

It is an element property by element.

- **matrix definita positiva**:

$$x^T Ax > 0 \quad \forall x \neq 0.$$

It is a property of the square form.

- **matrix ortogonale**:

$$Q^T Q = I.$$

Doesn't mean the matrix elements are"ortogonali": rows and columns are orthonormal.

- **matrix diagonalizzabile**does not necessarily mean symmetric.
- **matrix simmetrica**real is always diagonalized orthogonally.
- **matrix normale**it is always diagonally, but it is not necessarily Hermitian.
- **Semidefinita positiva**does not imply invertability.
- **Strettamente diagonalmente dominante**implies invertability, but not necessarily positive definiteness.



28. Block Matrices and the Schur Complement

Both

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

For a diagonal block matrix:

$$\det \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \det A \det D, \qquad \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix}.$$

A is invertible, Schur complement A and

$$S_D = D - CA^{-1}B.$$

Yes.

$$M = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & S_D \end{pmatrix}, \qquad \det M = \det A \det S_D.$$

AandS_D are invertible:

$$M^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BS_D^{-1}CA^{-1} & -A^{-1}BS_D^{-1} \\ -S_D^{-1}CA^{-1} & S_D^{-1} \end{pmatrix}.$$

For $M = M^*$:

$$M = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}, \quad A \succ 0,$$

$$M \succ 0 \iff D - B^*A^{-1}B \succ 0.$$

System block removal $M(x,y)^T = (f,g)^T$ produces

$$S_D y = g - CA^{-1}f, \qquad Ax = f - By.$$

28.1 Saddle-Point and KKT Matrices

$$K = \begin{pmatrix} H & A^* \\ A & 0 \end{pmatrix}.$$

$H = H^*$,K is Hermitian but generally indefinite. $H \succ 0$ and A has full rank for rows, K is not unusual. Schur's complement is'

$$S = -AH^{-1}A^* \prec 0.$$



29. LU Factorization of Rectangular Matrices

Both

$$A \in \mathbb{K}^{m\times n}.$$

29.1 LU without Pivoting

When gaussiane eliminations can be performed without row exchanges, a rectangular matrix admits a factorization of the type

$$A = LU.$$

$m \geq n$:

$$L \in \mathbb{K}^{m\times n}, \qquad U \in \mathbb{K}^{n\times n},$$

with L bottom trapezoidal and U superior triangular.

$m \leq n$:

$$L \in \mathbb{K}^{m\times m}, \qquad U \in \mathbb{K}^{m\times n},$$

with L lower triangular and U Top trapezoidal.

29.2 LU with Partial Pivoting

The norm form is

$$PA = LU,$$

where P is a matrix of permutation of the rows.

For rectangular plates:

- the partial pivoting exchange rows;
- the factorization proceeds up to $\min(m,n)$ steps;
- the numerical rank can emerge from the presence of small or null pivot.

29.3 LU with Full Pivoting

Factorization with full pivoting is

$$PAQ = LU,$$

where:

- P permutes the rows;

- Q permutes the columns.

At each step you choose as a pivot the maximum module element in the subblock still not factorized.

Advantages:

- greater robustness than partial pivoting;
- useful to reveal the rank.

Disadvantages

- higher cost in search of pivot;
- more permutations.

29.4 Rank-Revealing Form with Full Pivoting

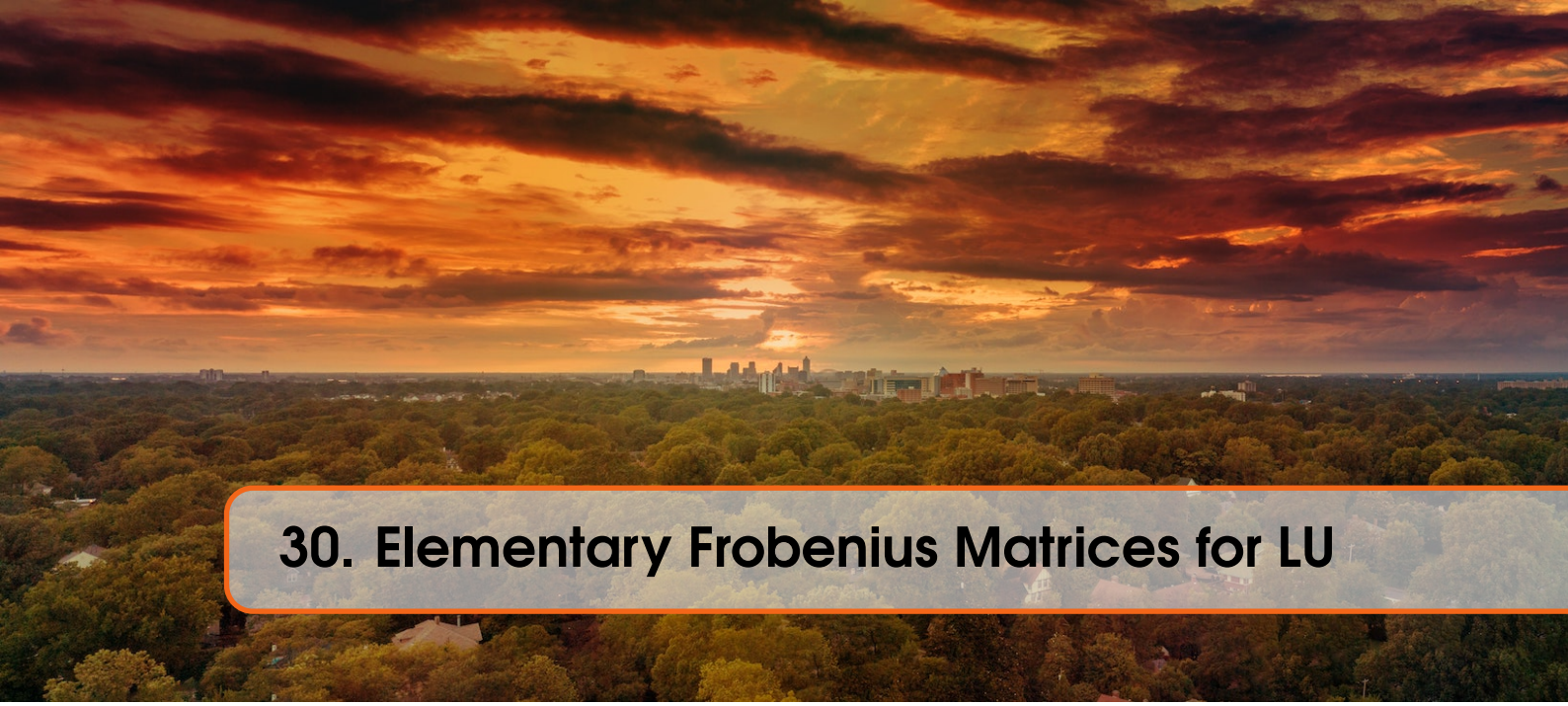
$\text{rank}(A) = r$, after appropriate permutations:

$$PAQ = \begin{pmatrix} L_{11} \\ L_{21} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \end{pmatrix},$$

where

$$L_{11} \in \mathbb{K}^{r \times r}, \qquad U_{11} \in \mathbb{K}^{r \times r}$$

They are not singular.



30. Elementary Frobenius Matrices for LU

An elementary Frobenius matrix associated with Gaussian elimination has the shape

$$F_k = I - \ell_k e_k^T,$$

where ℓ_k contains the multipliers used to undo the items under the column pivot k.

For example:

$$F_k = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & -\ell_{k+1,k} & 1 & \\ & & \vdots & & \ddots \\ & & -\ell_{n,k} & & 1 \end{pmatrix}.$$

Property:

- F_k is lower triangular;
- $\det(F_k) = 1$;
- F_k is invertible;
-

$$F_k^{-1} = I + \ell_k e_k^T.$$

Gaussian elimination can be written as

$$U = F_{n-1} \cdots F_2 F_1 A.$$

So:

$$A = LU, \qquad L = F_1^{-1} F_2^{-1} \cdots F_{n-1}^{-1}.$$

With pivoting:

$$PA = LU.$$



31. LDL^* Factorization

For hermitiane machines:

$$A = LDL^*.$$

For indefinite matrices, symmetric pivoting is used:

$$P^*AP = LDL^*,$$

with D diagonal blocks, typically with blocks 1×1 and 2×2. In the real case:

$$P^TAP = LDL^T.$$

It is particularly suitable for indefinite symmetric systems and KKT.



32. QR Factorization of Rectangular Matrices

Both

$$A \in \mathbb{K}^{m \times n}.$$

32.1 Tall Matrix: $m \geq n$

A has full rank for columns:

$$\operatorname{rank}(A) = n.$$

Reduced QR:

$$A = QR,$$

with

$$Q \in \mathbb{K}^{m \times n}, \qquad Q^* Q = I_n,$$

and

$$R \in \mathbb{K}^{n \times n}$$

superior and not singular triangular.

Full QR:

$$A = \widehat{Q} \begin{pmatrix} R \\ 0 \end{pmatrix},$$

where

$$\widehat{Q} \in \mathbb{K}^{m \times m}$$

is unitary/orthagonal.

32.2 Wide Matrix: $m \leq n$

A norm QR produces:

$$A = QR,$$

with

$$Q \in \mathbb{K}^{m \times m}, \qquad R \in \mathbb{K}^{m \times n},$$

where R is superior trapezoidal.

A has full rank for rows:

$$\text{rank}(A) = m.$$

32.3 QR with Column Pivoting

The shape is

$$AP = QR,$$

where P is a matrix of permutation of the columns.

It is used for:

- improve stability;
- reveal the rank;
- deal with problems at the least squares-deficient.

$\text{rank}(A) = r$, ideally:

$$AP = Q \begin{pmatrix} R_{11} & R_{12} \\ 0 & 0 \end{pmatrix},$$

with

$$R_{11} \in \mathbb{K}^{r \times r}$$

not singular.



33. LQ, RRQR, and COD Factorizations

33.1 LQ Factorization

For $A \in \mathbb{K}^{m\times n}$ with $m \leq n$:

$$A = LQ.$$

A has full rank for rows, in reduced form $L \in \mathbb{K}^{m\times m}$ is not unusual and lower triangular $Q_1 Q_1^* = I_m$. LQ gets by applying QR to A^*.

33.2 Rank-Revealing QR

$$AP = QR = Q\begin{pmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{pmatrix}.$$

For numerical rank r, good RRQR makes $R_{11} \in \mathbb{K}^{r\times r}$ well conditioned and R_{22} Baby.

33.3 Complete Orthogonal Decomposition

For $\text{rank}(A) = r$:

$$A = Q\begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} Z^*,$$

with Q, Z Units and $T \in \mathbb{K}^{r\times r}$ triangular not singular. It is useful for numerical rank, least squares-deficient and minimum-norm solutions.



34. SVD: Forms and Advanced Properties

Both $A \in \mathbb{K}^{m\times n}$, $\operatorname{rank}(A) = r$. Complete SVD:

$$A = U\Sigma V^*.$$

SVD compact:

$$A = U_r\Sigma_r V_r^*, \qquad \Sigma_r = \operatorname{diag}(\sigma_1,\dots,\sigma_r), \quad \sigma_r > 0.$$

From SVD:

$$\operatorname{range}(A) = \operatorname{span}(u_1,\dots,u_r), \qquad \ker(A^*) = \operatorname{span}(u_{r+1},\dots,u_m),$$
$$\operatorname{range}(A^*) = \operatorname{span}(v_1,\dots,v_r), \qquad \ker(A) = \operatorname{span}(v_{r+1},\dots,v_n).$$

The pseudoinverse is

$$A^\dagger = V_r\Sigma_r^{-1}U_r^*.$$

34.1 Eckart–Young–Mirsky Theorem

For $k < r$:

$$A_k = U_k\Sigma_k V_k^*$$

is a better approximation of rank to the most «k, with

$$\|A - A_k\|_2 = \sigma_{k+1}, \qquad \|A - A_k\|_F = \left(\sum_{j=k+1}^{r}\sigma_j^2\right)^{1/2}.$$



35. Table of Main Factorizations

Factorization	**matrix**	**Form**	**Use**
L	General	$PAQ = LU$	systems, elimination, rank
Cholesky	SPD/HPD	$A = LL^*$	positive defined systems
LDL^*	Hermitian	$P^*AP = LDL^*$	indefinite, KKT
QR	above all $m \geq n$	$A = QR$	least squares
L	above all $m \leq n$	$A = LQ$	minimum-norm
RRQR	General	$AP = QR$	numerical rank
COD	General	$A = Q\left(\begin{smallmatrix} T & 0 \\ 0 & 0 \end{smallmatrix}\right)Z^*$	deficient rank
SVD	General	$A = U\Sigma V^*$	rank, pseudoinverse, LS
Schur	square	$A = QTQ^*$	eigenvalues
Jordan	square	$A = VJV^{-1}$	theory



36. Generalized Inverses and the Pseudoinverse

36.1 Left Inverse

For

$$A \in \mathbb{K}^{m \times n}, \qquad m \geq n,$$

a matrix

$$L \in \mathbb{K}^{n \times m}$$

is a left inverse if

$$LA = I_n.$$

It exists if and only if:

$$\operatorname{rank}(A) = n.$$

A canonical choice:

$$L = (A^* A)^{-1} A^*.$$

36.2 Right Inverse

For

$$A \in \mathbb{K}^{m \times n}, \qquad m \leq n,$$

a matrix

$$R \in \mathbb{K}^{n \times m}$$

is a right inverse if

$$AR = I_m.$$

It exists if and only if:

$$\operatorname{rank}(A) = m.$$

A canonical choice:

$$R = A^* (A A^*)^{-1}.$$

36.3 Generalized Inverse

A matrix G is a generalized inverse of A if

$$AGA = A.$$

In general it is not unique.

36.4 Moore–Penrose Pseudoinverse

Moore's pseudoinverse–Penrose

$$A \in \mathbb{K}^{m \times n}$$

is the only matrix

$$A^{\dagger} \in \mathbb{K}^{n \times m}$$

that satisfies:

$$AA^{\dagger}A = A,$$

$$A^{\dagger}AA^{\dagger} = A^{\dagger},$$

$$(AA^{\dagger})^{*} = AA^{\dagger},$$

$$(A^{\dagger}A)^{*} = A^{\dagger}A.$$

36.4.1 Minimum Properties

For every $b \in \mathbb{K}^{m}$, the vector

$$x^{\dagger} = A^{\dagger}b$$

is the unique minimum-Euclidean-norm least-squares solution:

$$x^{\dagger} = \operatorname*{argmin}_{x} \left\{ \|x\|_2 : x \in \operatorname*{argmin}_{z} \|Az - b\|_2 \right\}.$$

In particular, if $Ax = b$ is consistent, then $A^{\dagger}b$ is the unique solution of minimum norm. Moreover,

$$AA^{\dagger}b = \operatorname*{argmin}_{y \in \operatorname{range}(A)} \|b - y\|_2,$$

and the minimum residual is orthogonal to the range of A:

$$b - AA^{\dagger}b \in \ker(A^{*}).$$

Among all generalized inverses G satisfying $AGA = A$, the Moore–Penrose pseudoinverse is the unique one with minimum Frobenius norm:

$$A^{\dagger} = \operatorname*{argmin}_{G : AGA = A} \|G\|_F.$$

If A has full row rank, $A^{\dagger}$ is the minimum-Frobenius-norm right inverse; if A has full column rank, it is the minimum-Frobenius-norm left inverse.

36.5 Pseudoinverse via the SVD

$$A = U\Sigma V^*,$$

then

$$A^\dagger = V\Sigma^\dagger U^*,$$

where

$$\Sigma^\dagger$$

you get inverting only the singular values not null.

$$\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_r, 0, \dots),$$

then

$$\Sigma^\dagger = \operatorname{diag}\left(\frac{1}{\sigma_1}, \dots, \frac{1}{\sigma_r}, 0, \dots\right)$$

with the transposed dimensions.

36.6 Pseudoinverse via Two QR Factorizations

Let $A \in \mathbb{K}^{m\times n}$ have rank r. First compute a rank-revealing QR factorization with column pivoting and retain only the nonzero part:

$$A\Pi = Q_1 R,$$

where $Q_1 \in \mathbb{K}^{m\times r}$ has orthonormal columns and $R \in \mathbb{K}^{r\times n}$ has full row rank. Then compute a thin QR factorization of R^*:

$$R^* = ZT,$$

where $Z \in \mathbb{K}^{n\times r}$ has orthonormal columns and $T \in \mathbb{K}^{r\times r}$ is nonsingular and upper triangular. Since $R = T^*Z^*$,

$$R^\dagger = ZT^{-*},$$

and therefore

$$\boxed{A^\dagger = \Pi Z T^{-*} Q_1^*.}$$

No inverse needs to be formed explicitly: solve the triangular system

$$T^*Y = Q_1^*$$

and assemble $A^\dagger = \Pi ZY$. Thus a general rank-r pseudoinverse can be computed with two QR factorizations and triangular solves, without forming the normal equations.

36.7 Special Cases

A is square and invertible:

$$A^\dagger = A^{-1}.$$

A has full rank for columns:

$$A^\dagger = (A^*A)^{-1}A^*.$$

A has full rank for rows:

$$A^\dagger = A^*(AA^*)^{-1}.$$

36.8 Projectors Associated with the Pseudoinverse

$$AA^\dagger$$

is the orthogonal projector on

$$\text{range}(A).$$

$$A^\dagger A$$

is the orthogonal projector on

$$\text{range}(A^*) = (\ker A)^\perp.$$

$$I - AA^\dagger$$

is the orthogonal projector on

$$\ker(A^*).$$

$$I - A^\dagger A$$

is the orthogonal projector on

$$\ker(A).$$

36.9 Least Squares and the Pseudoinverse

The minimum norm solution of the problem

$$\min_x \|Ax - b\|_2$$

is:

$$x^\dagger = A^\dagger b.$$

All solutions at the smallest squares are:

$$x = A^\dagger b + (I - A^\dagger A)z, \qquad z \in \mathbb{K}^n.$$

The minimum residue is:

$$r_\star = b - AA^\dagger b.$$

Projection b up $\text{range}(A)$ is:

$$AA^\dagger b.$$



37. Pseudoinverse: Identities and Decompositions

In addition to Moore–Penrose conditions:

$$(A^\dagger)^\dagger = A, \qquad (A^*)^\dagger = (A^\dagger)^*,$$
$$\operatorname{range}(A^\dagger) = \operatorname{range}(A^*), \qquad \ker(A^\dagger) = \ker(A^*).$$

In general:

$$(AB)^\dagger eB^\dagger A^\dagger.$$

The matrices

$$AA^\dagger, \qquad A^\dagger A$$

are respectively the orthogonal projectors on $\operatorname{range}(A)$ and $\operatorname{range}(A^*)$.



38. Least-Squares Problems

Both

$$A \in \mathbb{K}^{m \times n}, \qquad b \in \mathbb{K}^m.$$

The problem at the least squares is

$$\min_{x \in \mathbb{K}^n} \|Ax - b\|_2.$$

38.1 Normal Equations

Every solution to the least squares meets:

$$A^*(Ax - b) = 0,$$

i.e.

$$A^*Ax = A^*b.$$

Geometric interpretation:

$$Ax$$

is the orthogonal projection of b up

$$\text{range}(A).$$

38.2 Tall Full-Column-Rank Case

$$m \geq n, \qquad \text{rank}(A) = n,$$

the solution to the least squares is unique:

$$x = (A^*A)^{-1}A^*b.$$

Using reduced QR:

$$A = QR,$$

you have

$$Rx = Q^*b.$$

38.3 Tall Rank-Deficient Case

$$m \geq n, \qquad \operatorname{rank}(A) = r < n,$$

solutions to the least squares are not in general unique.

The whole of the solutions is close:

$$x = x_0 + z, \qquad z \in \ker(A).$$

Among all the solutions to the least squares there is a single minimum Euclidean norm solution:

$$x^\dagger = A^\dagger b.$$

38.4 Wide Full-Row-Rank Case

$$m < n, \qquad \operatorname{rank}(A) = m,$$

the system

$$Ax = b$$

is compatible for each b.

There are endless solutions.

The minimum norm solution is:

$$x^\dagger = A^*(AA^*)^{-1}b.$$

38.5 Wide Rank-Deficient Case

$$m < n, \qquad \operatorname{rank}(A) = r < m,$$

the system can be incompatible.

You search:

$$\min_x \|Ax - b\|_2.$$

The minimum norm solution between all solutions at the least squares is:

$$x^\dagger = A^\dagger b.$$

38.6 Characterization of Least-Squares Solutions

$x_\star$ is solution to the least squares if and only if:

$$A^*(Ax_\star - b) = 0.$$

Equivalently:

$$b - Ax_\star \perp \operatorname{range}(A).$$

38.7 Numerical Choice of Method

A has full rank for columns, normal equations meet

$$\kappa_2(A^*A) = kappa_2(A)^2.$$

QR avoids the conditioning square due to the formation of A^*A; SVD and COD are particularly indicated in rank-deficient issues. For full row and natural matrices use LQ or QR A^*.



39. Condition Number

For an invertible square matrix:

$$\underset{p}{\text{cond}}(A) = \|A\|_p \|A^{-1}\|_p.$$

norm 2:

$$\underset{2}{\text{cond}}(A) = \frac{\sigma_{\max}(A)}{\sigma_{\min}(A)}.$$

For a full-scale rectangular matrix:

$$\underset{2}{\text{cond}}(A) = \frac{\sigma_{\max}(A)}{\sigma_{\min}(A)},$$

where $\sigma_{\min}$ is the smallest not null singular value.

In terms of pseudoinverse:

$$\underset{2}{\text{cond}}(A) = \|A\|_2 \|A^{\dagger}\|_2.$$

For an orthogonal/unitarian matrix:

$$\underset{2}{\text{cond}}(A) = 1.$$



40. Perturbations, Residuals, and Numerical Stability

For $Ax = b$, A not singular:

$$\kappa(A) = \|A\| \, \|A^{-1}\|.$$

If it is disturbed only b:

$$\frac{\|\delta x\|}{\|x\|} \leq \kappa(A) \frac{\|\delta b\|}{\|b\|}.$$

The residue of an approximate solution $\widehat{x}$ and

$$r = b - A\widehat{x}.$$

A small residue does not necessarily imply small forward error if A is bad condition. The backward error measures the minimum disruption of data for which $\widehat{x}$ is an exact solution. Conditioning is a property of the problem; stability is a property of the algorithm.



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