Mathematical Analysis Dictionary

Definitions, Theorems, and Essential Formulas

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1. Notation and Preliminary Concepts

Purpose and Scope

This is not a set of lecture notes. It is a compact dictionary of definitions and results used in the accompanying courses. If you want to study any of these topics, you should follow a course or consult a textbook devoted to the subject. This dictionary is intended for readers who already know, or have previously studied, the material and need a concise reference whenever a reminder is useful.

In the following, unless otherwise stated,

$$f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m.$$

The Euclidean norm is indicated with $\|\cdot\|$.

For $x_0 \in \mathbb{R}^n$:

$$B_r(x_0) = \{x : \|x - x_0\| < r\}, \qquad \overline{B}_r(x_0) = \{x : \|x - x_0\| \leq r\}.$$

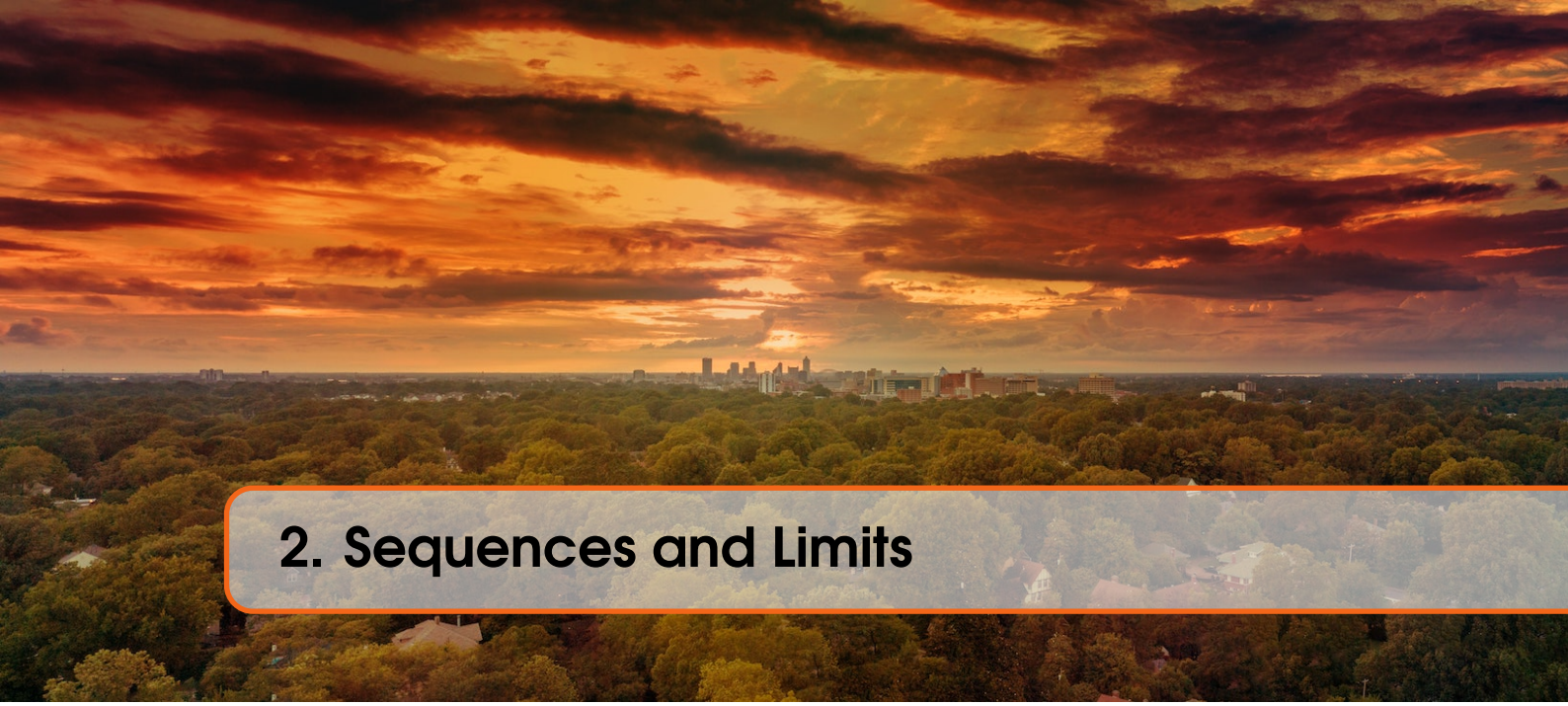
A point x_0 is accumulation for A if every $B_r(x_0)$ contains points of A different from x_0.

1.1 Interior, Closure, and Boundary

$$\operatorname{int}(A) = \{x : \exists r > 0,\ B_r(x) \subset A\},$$

$$\operatorname{cl}(A) = \text{smallest closed set containing } A,$$

$$\partial A = \operatorname{cl}(A) \setminus \operatorname{int}(A).$$



2. Sequences and Limits

2.1 Convergence of Sequences

A sequence $x_k \in \mathbb{R}^n$ converging x if

$$\forall \varepsilon > 0 \; \exists N : \; k \geq N \Rightarrow \|x_k - x\| < \varepsilon.$$

It is written

$$x_k \to x.$$

In $\mathbb{R}^n$:

$$x_k \to x \iff (x_k)_i \to x_i \quad \forall i.$$

2.2 Cauchy Criterion

In $\mathbb{R}^n$:

$$x_k \text{ converges} \iff \forall \varepsilon > 0 \; \exists N : \; p,q \geq N \Rightarrow \|x_p - x_q\| < \varepsilon.$$

2.3 Bolzano–Weierstrass Theorem

Every bounded sequence in $\mathbb{R}^n$ admits a converging subsequence.

2.4 Limit of a Function

We say

$$\lim_{x \to x_0} f(x) = L$$

if

$$\forall \varepsilon > 0 \; \exists \delta > 0 : 0 < \|x - x_0\| < \delta \Rightarrow \|f(x) - L\| < \varepsilon.$$

2.5 Sequential Characterization

x_0 is an accumulation point of A:

$$\lim_{x \to x_0} f(x) = L \iff f(x_k) \to L$$

for every sequence $x_k \in A \setminus \{x_0\}$ with $x_k \to x_0$.

2.6 Uniqueness of the Limit

If the limit exists, it is unique.

2.7 Algebra of Limits

$f \to a$ and $g \to b$:

$$f+g \to a+b, \qquad fg \to ab,$$
$$\frac{f}{g} \to \frac{a}{b} \quad \text{if } b \neq 0 \text{ and } g \neq 0 \text{ locally.}$$

For vector functions the limit is component by component.

2.8 Iterated Limits and Joint Limits

In general:

$$\lim_{x \to x_0} \lim_{y \to y_0} f(x,y)$$

and

$$\lim_{(x,y) \to (x_0,y_0)} f(x,y)$$

are different concepts.

If the joint limit exists, every limit along an admissible curve or sequence coincide with it. Limits along individual matching curves are not sufficient, in general, to ensure the existence of the joint limit.



3. Continuity

f is continuous $x_0 \in A$ if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Equivalently:

$$\forall \varepsilon > 0 \; \exists \delta > 0 : \|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \varepsilon.$$

3.1 Sequential Characterization

$$f \text{ continuous at } x_0 \iff x_k \to x_0 \Rightarrow f(x_k) \to f(x_0).$$

3.2 Operations on Continuous Functions

Sum, product, composition and quotient with nonzero denominator of continuous functions are continuous.

3.3 Uniform Continuity

$f : A \to \mathbb{R}^m$ is uniformly continuous if

$$\forall \varepsilon > 0 \; \exists \delta > 0 : \; \|x - y\| < \delta \Rightarrow \|f(x) - f(y)\| < \varepsilon$$

for each $x, y \in A$.

Conceptual difference:

- continuity: δ can depend on the point;
- uniform continuity: δ depends only on ε.

3.4 Heine–Cantor Theorem

$K \subset \mathbb{R}^n$ is compact and $f : K \to \mathbb{R}^m$ it is continuous, then f is uniformly continuous.

3.5 Weierstrass Theorem

$K \subset \mathbb{R}^n$ is compact and $f : K \to \mathbb{R}$ it is continuous, then f is limited and attains its maximum and minimum.



4. Lipschitz and Hölder Continuity

4.1 Lipschitz Continuity

$f : A \to \mathbb{R}^m$ is Lipschitz if it exists $L \geq 0$ such that

$$\|f(x) - f(y)\| \leq L\|x - y\| \qquad \forall x, y \in A.$$

You can define

$$\mathrm{Lip}(f;A) = \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{\|x - y\|}.$$

Each Lipschitz function is uniformly continuous.

4.2 Local Lipschitz Continuity

f is locally Lipschitz if every point has a neighborhood on which f is Lipschitz.

4.3 A Sufficient Derivative Condition

A It is convex, $f \in C^1(A)$ and

$$\sup_{x \in A} \|Df(x)\| \leq L,$$

then

$$\|f(x) - f(y)\| \leq L\|x - y\|.$$

In a variable:

$$|f'(x)| \leq L \quad \Rightarrow \quad |f(x) - f(y)| \leq L|x - y|.$$

4.4 Hölder Continuity

For $0 < \alpha \leq 1$, f is Hölder of exponent α if

$$\|f(x) - f(y)\| \leq C\|x - y\|^{\alpha}.$$

The case $\alpha = 1$ coincides with Lipschitz.



5. Derivatives in One Variable

For $f : I \subset \mathbb{R} \to \mathbb{R}^m$:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}.$$

Equivalently:

$$f(x_0+h) = f(x_0) + f'(x_0)h + o(h).$$

5.1 Differentiability Implies Continuity

$$f \text{ differentiable at } x_0 \implies f \text{ continuous at } x_0.$$

The converse is false.

5.2 Differentiation Rules

$$(f+g)' = f' + g',$$

$$(fg)' = f'g + fg',$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2},$$

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

5.3 Rolle's Theorem

$f \in C([a,b])$, differentiable on (a,b) and $f(a) = f(b)$, then there exists $\xi \in (a,b)$ with

$$f'(\xi) = 0.$$

5.4 Lagrange's Mean Value Theorem

$f \in C([a,b])$ and is differentiable on (a,b), then there exists $\xi \in (a,b)$ such that

$$f(b) - f(a) = f'(\xi)(b-a).$$

5.5 Cauchy's Mean Value Theorem

Under the usual assumptions:

$$[f(b)-f(a)]g'(\xi)=[g(b)-g(a)]f'(\xi).$$



6. Partial and Directional Derivatives

6.1 Partial Derivative

For $f : \mathbb{R}^n \to \mathbb{R}^m$:

$$\frac{\partial f}{\partial x_i}(x) = \lim_{h \to 0} \frac{f(x + he_i) - f(x)}{h}.$$

6.2 Directional Derivative

In the direction v:

$$D_v f(x) = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}.$$

For $f : \mathbb{R}^n \to \mathbb{R}$ differentiable:

$$D_v f(x) = \nabla f(x)^T v.$$

$\|v\| = 1$:

$$|D_v f(x)| \leq \|\nabla f(x)\|,$$

with maximum in the direction of gradient.

6.3 Existence of Directional Derivatives

The existence of all directional derivatives at one point does not generally imply differentiability at the point.



7. Differentiability in Several Variables

7.1 Fréchet Derivative

$f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable in x if there is a linear application

$$Df(x) : \mathbb{R}^n \to \mathbb{R}^m$$

such that

$$f(x+h) = f(x) + Df(x)h + o(\|h\|).$$

Equivalently:

$$\lim_{h\to 0} \frac{\|f(x+h) - f(x) - Df(x)h\|}{\|h\|} = 0.$$

$Df(x)$ is unique.

7.2 Jacobian

For $f = (f_1, \ldots, f_m)$:

$$Df(x) = J_f(x) = \begin{pmatrix} \partial_1 f_1 & \cdots & \partial_n f_1 \\ \vdots & & \vdots \\ \partial_1 f_m & \cdots & \partial_n f_m \end{pmatrix}.$$

For $m = 1$:

$$Df(x) = \nabla f(x)^T.$$

7.3 Differentiability and Partial Derivatives

f is differentiable, then all partial derivatives exist.

The converse is false in general.

Classic enough condition:

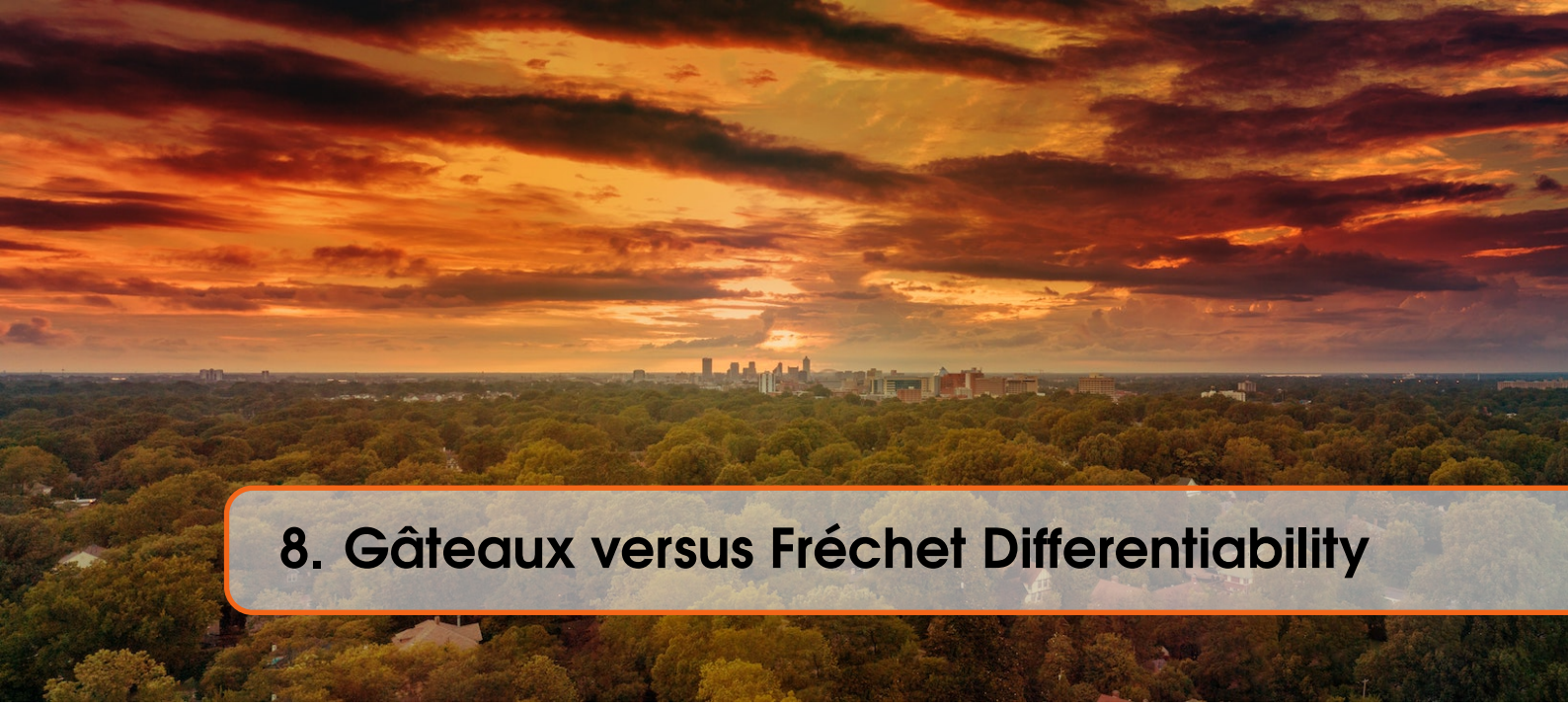
$\partial_i f_j$ exist in a neighborhood and are continuous at $x_0 \implies f$ differentiable at x_0.

In particular:

$$f \in C^1 \implies f \text{ differentiable.}$$

7.4 Differentiability Implies Continuity

$$f \text{ differentiable at } x_0 \Longrightarrow f \text{ continuous at } x_0.$$



8. Gâteaux versus Fréchet Differentiability

8.1 Gâteaux Derivative

The derivative of Gâteaux f in x in the direction h It is

$$D_G f(x)[h] = \lim_{t \to 0} \frac{f(x+th) - f(x)}{t}.$$

If it exists for each h and is linear in h, we talk about Gâteaux's differentiability.

8.2 Relationships

Fréchet differentiable $\Longrightarrow$ Gâteaux differentiable.

In finite dimensions, under appropriate assumptions of continuity of the derivative, the differentiability of Gâteaux can imply that of Fréchet.

The only existence of directional derivatives does not guarantee neither linearity nor differentiability of Fréchet.



9. Chain Rule and Derivatives of Compositions

$$f : \mathbb{R}^n \to \mathbb{R}^m, \qquad g : \mathbb{R}^m \to \mathbb{R}^p$$

are differentiable:

$$D(g \circ f)(x) = Dg(f(x))Df(x).$$

For $g : \mathbb{R}^m \to \mathbb{R}$:

$$\nabla(g \circ f)(x) = Df(x)^T \nabla g(f(x)).$$

$x = x(t)$:

$$\frac{d}{dt} f(x(t)) = Df(x(t))x'(t).$$

For $f : \mathbb{R}^n \to \mathbb{R}$:

$$\frac{d}{dt} f(x(t)) = \nabla f(x(t))^T x'(t).$$



10. Second Derivatives and the Hessian

10.1 Hessian

For $f : \mathbb{R}^n \to \mathbb{R}$ twice differentiable:

$$\operatorname{Hess} f(x) = \nabla^2 f(x) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{i,j}.$$

10.2 Schwarz–Clairaut Theorem

If the second mixed derivatives are continuous in a neighborhood:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

So Hessian is symmetric.

10.3 Second Directional Derivative

$f \in C^2$:

$$D_v^2 f(x) = v^T \operatorname{Hess} f(x) v.$$

10.4 Second Derivative of a Composition along a Curve

$x = x(t)$ and $f : \mathbb{R}^n \to \mathbb{R}$:

$$\frac{d^2}{dt^2} f(x(t)) = x'(t)^T \operatorname{Hess} f(x(t)) x'(t) + \nabla f(x(t))^T x''(t).$$



11. Regularity Classes

$$C^0 = \text{continuous functions},$$

$$C^1 = \text{functions with continuous first derivatives},$$

$$C^k = \text{functions with continuous derivatives up to order } k \text{ continuous},$$

$$C^\infty = \bigcap_{k \geq 0} C^k.$$

An analytical function locally admits a development in series of powers converging to function. In general:

$$\text{analytic} \Longrightarrow C^\infty,$$

but vice versa is false.



12. Asymptotic Notation: Little o and Big O

12.1 Little o

For $x \to x_0$:

$$f(x) = o(g(x))$$

if

$$\frac{f(x)}{g(x)} \to 0$$

when the report is defined locally.

Equivalently:

$$f = o(g) \iff \forall \varepsilon > 0 \; \exists \delta > 0 : |f(x)| \leq \varepsilon |g(x)|$$

for x sufficiently close to x_0.

12.2 Big O

$$f(x) = O(g(x))$$

if they exist $C > 0$ and a neighborhood of x_0 such as

$$|f(x)| \leq C|g(x)|.$$

12.3 Relationship between o and O

$$f = o(g) \Longrightarrow f = O(g).$$

The converse is false.

12.4 Asymptotic Equivalence

$$f(x) \sim g(x) \quad \iff \quad \frac{f(x)}{g(x)} \to 1.$$

Equivalently:

$$f = g + o(g).$$

12.5 Θ Notation

$$f = \Theta(g)$$

if

$$c|g(x)| \leq |f(x)| \leq C|g(x)|$$

locally, for some $c, C > 0$.

12.6 Algebraic Rules

$f = o(g)$ and $h = O(k)$:

$$fh = o(gk).$$

$f_1 = o(g)$ and $f_2 = o(g)$:

$$f_1 + f_2 = o(g).$$

$f = O(g)$ and $h = O(k)$:

$$fh = O(gk).$$

$f = o(1)$:

$$1 + f = 1 + o(1).$$

12.7 Vector-Valued Version

For $f : \mathbb{R}^n \to \mathbb{R}^m$:

$$f(h) = o(\|h\|^k)$$

means

$$\frac{\|f(h)\|}{\|h\|^k} \to 0.$$

Similarly:

$$f(h) = O(\|h\|^k) \iff \|f(h)\| \leq C\|h\|^k.$$



13. Elementary Asymptotic Expansions

For $x \to 0$:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4),$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4),$$

$$\sin x = x - \frac{x^3}{6} + O(x^5),$$

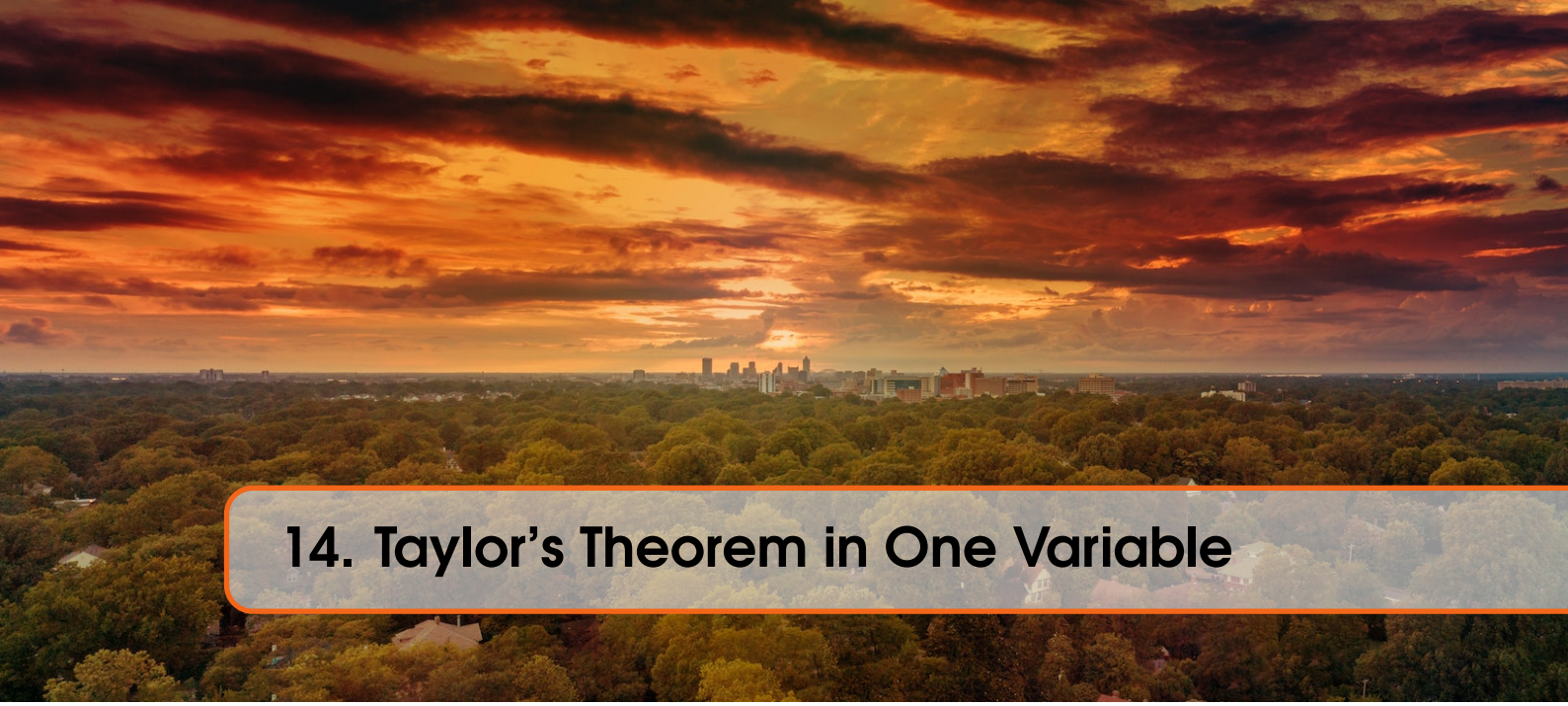
$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6),$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + O(x^3).$$

In particular:

$$\sin x \sim x, \qquad \tan x \sim x, \qquad 1 - \cos x \sim \frac{x^2}{2},$$

$$e^x - 1 \sim x, \qquad \log(1+x) \sim x.$$



14. Taylor's Theorem in One Variable

Both f sufficiently regular near to x_0. Taylor's polynomial of order p It is

$$T_p(x;x_0) = \sum_{k=0}^{p} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$

14.1 Peano Remainder

f It is p times differentiable in an appropriate sense:

$$f(x) = \sum_{k=0}^{p} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + o(|x-x_0|^p).$$

14.2 Lagrange Remainder

$f \in C^{p+1}$ between x_0 and x:

$$f(x) = T_p(x;x_0) + \frac{f^{(p+1)}(\xi)}{(p+1)!}(x-x_0)^{p+1}$$

for some ξ between x_0 and x.

14.3 Integral Remainder

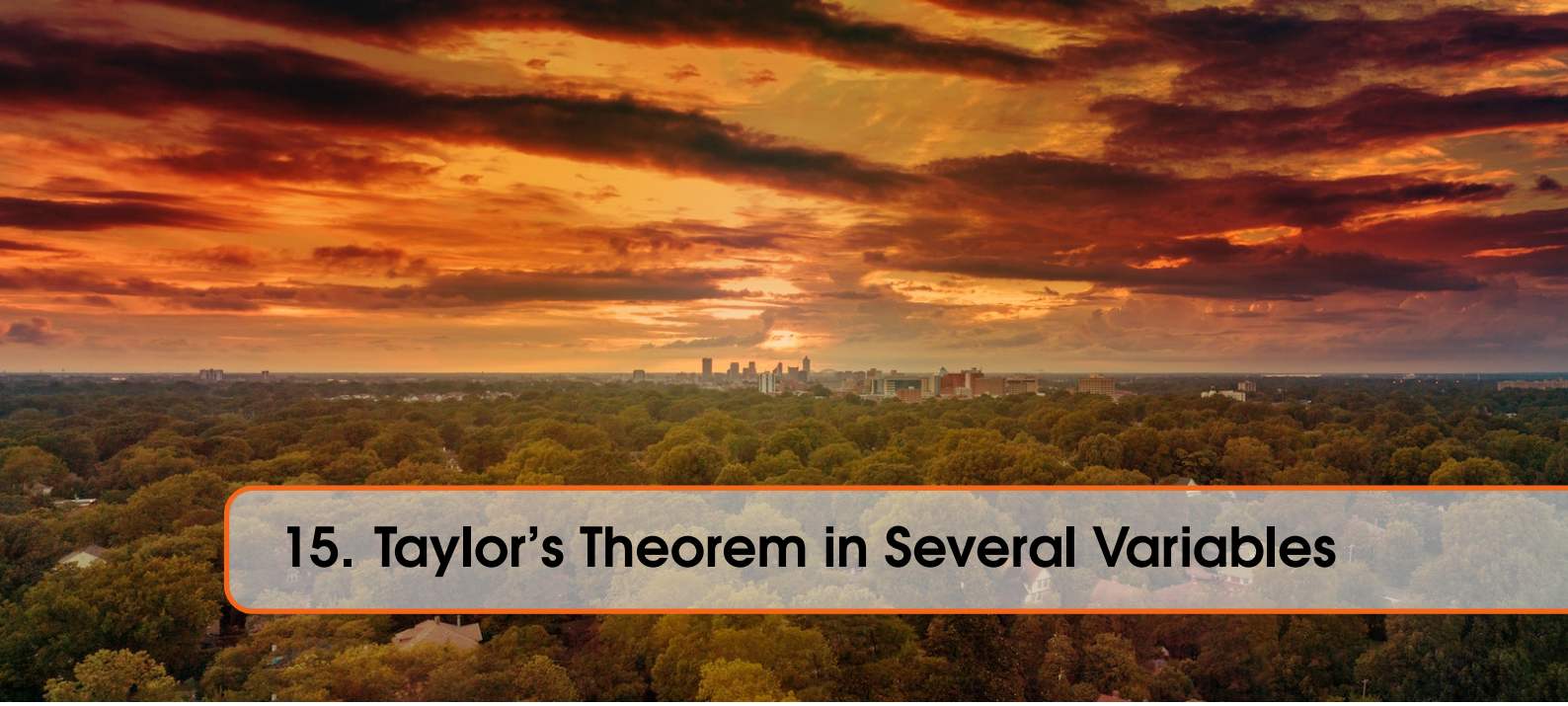
$f^{(p+1)}$ is continuous:

$$f(x) = T_p(x;x_0) + \frac{1}{p!}\int_{x_0}^{x} f^{(p+1)}(t)(x-t)^p\,dt.$$

14.4 Big-O Form

$f^{(p+1)}$ is locally limited:

$$f(x) = T_p(x;x_0) + O(|x-x_0|^{p+1}).$$



15. Taylor's Theorem in Several Variables

Both $f : \mathbb{R}^n \to \mathbb{R}$. With $h = x - x_0$:

15.1 First Order

f is differentiable:

$$f(x_0 + h) = f(x_0) + \nabla f(x_0)^T h + o(\|h\|).$$

Df is locally Lipschitz:

$$f(x_0 + h) = f(x_0) + \nabla f(x_0)^T h + O(\|h\|^2).$$

15.2 Second Order

f is twice differentiable with sufficient regularity:

$$f(x_0 + h) = f(x_0) + \nabla f(x_0)^T h + \frac{1}{2} h^T \operatorname{Hess} f(x_0) h + o(\|h\|^2).$$

$\operatorname{Hess} f$ is locally Lipschitz:

$$f(x_0 + h) = f(x_0) + \nabla f(x_0)^T h + \frac{1}{2} h^T \operatorname{Hess} f(x_0) h + O(\|h\|^3).$$

15.3 Form along a Line Segment

Defining

$$\phi(t) = f(x_0 + th),$$

Taylor multidimensional is reduced to Taylor 1D ϕ. In particular:

$$\phi'(t) = \nabla f(x_0 + th)^T h,$$
$$\phi''(t) = h^T \operatorname{Hess} f(x_0 + th) h.$$

15.4 Second-Order Lagrange Remainder

$f \in C^2$ the segment between x_0 and $x_0 + h$, exist $\theta \in (0, 1)$ that the

$$f(x_0 + h) = f(x_0) + \nabla f(x_0)^T h + \frac{1}{2} h^T \operatorname{Hess} f(x_0 + \theta h) h.$$

15.5 Multi-Index Form

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$:

$$|\alpha| = \sum_i \alpha_i, \qquad \alpha! = \prod_i \alpha_i!,$$

$$h^\alpha = \prod_i h_i^{\alpha_i}, \qquad D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

Then formally:

$$f(x_0 + h) = \sum_{|\alpha| \leq p} \frac{D^\alpha f(x_0)}{\alpha!} h^\alpha + R_p(h).$$

With appropriate hypotheses:

$$R_p(h) = o(\|h\|^p),$$

or, with a more locally limited derivative,

$$R_p(h) = O(\|h\|^{p+1}).$$



16. Mean Value Theorem in Several Variables

For $f : \mathbb{R}^n \to \mathbb{R}$, if the segment

$$[x,y] = \{x + t(y-x) : 0 \le t \le 1\}$$

is contained in the domain and f is differentiable, exists ξ on the segment

$$f(y) - f(x) = \nabla f(\xi)^T (y-x).$$

For $f : \mathbb{R}^n \to \mathbb{R}^m$ there is generally no analogue with a single point ξ valid simultaneously for all components. The estimate is:

$$\|f(y) - f(x)\| \le \sup_{z \in [x,y]} \|Df(z)\| \, \|y-x\|.$$



17. Functions with Lipschitz Gradient

$f : \mathbb{R}^n \to \mathbb{R}$ It is L-smooth se

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|.$$

$f \in C^2$ on a convex set:

$$\|\operatorname{Hess} f(x)\| \leq L \quad \forall x$$

implies ∇f It is L-Lipschitz.

17.1 Descent Lemma

∇f It is L-Lipschitz:

$$f(y) \leq f(x) + \nabla f(x)^T (y - x) + \frac{L}{2}\|y - x\|^2.$$

More precisely:

$$|f(y) - f(x) - \nabla f(x)^T (y - x)| \leq \frac{L}{2}\|y - x\|^2.$$



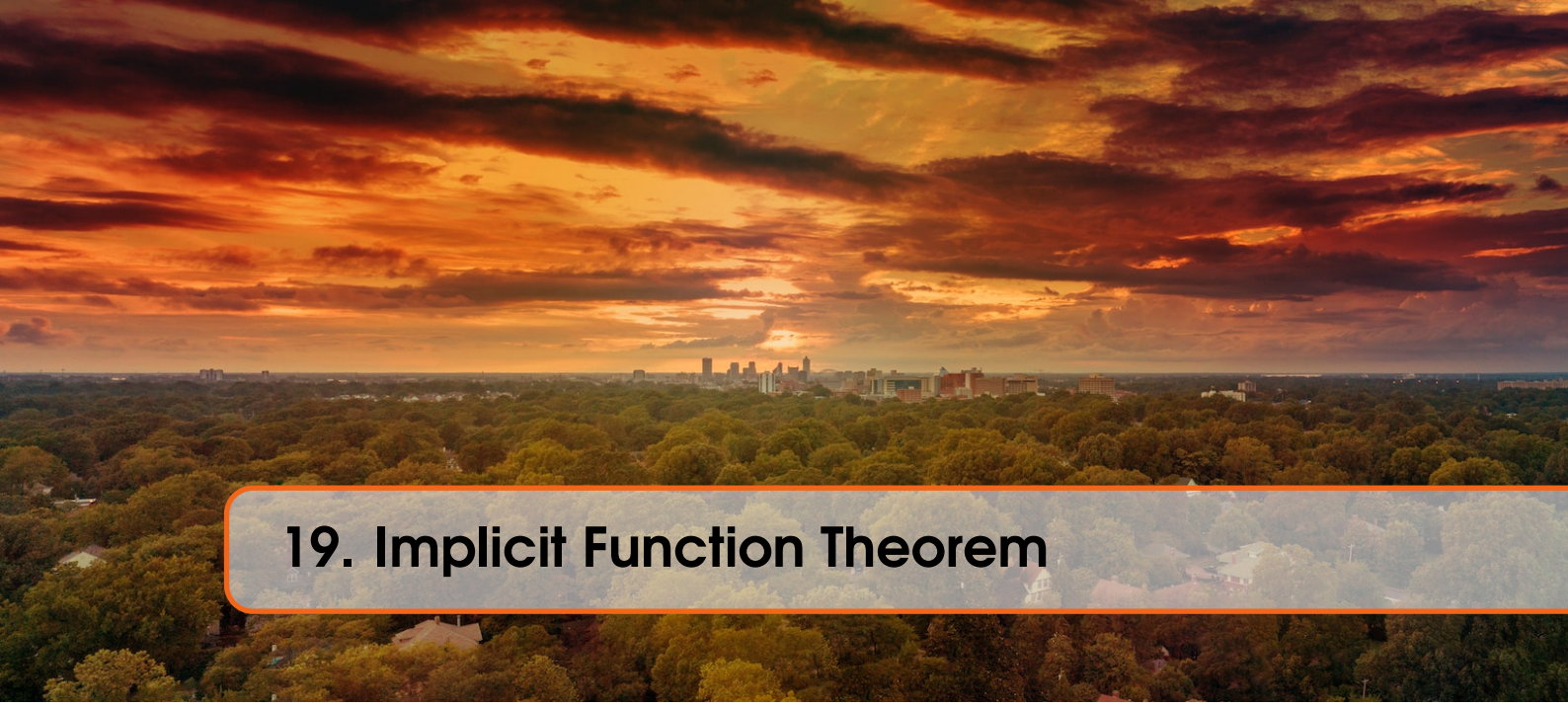
18. Inverse Function Theorem

Both $f : \mathbb{R}^n \to \mathbb{R}^n$ class C^1 in a neighborhood of x_0.

$$\det Df(x_0) \neq 0,$$

then f is locally invertible near x_0 and the inverse is C^1. In addition:

$$D(f^{-1})(f(x_0)) = Df(x_0)^{-1}.$$



19. Implicit Function Theorem

Both

$$F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$$

class C^1, with

$$F(x_0, y_0) = 0.$$

$$D_y F(x_0, y_0)$$

is invertible, then locally there is a function

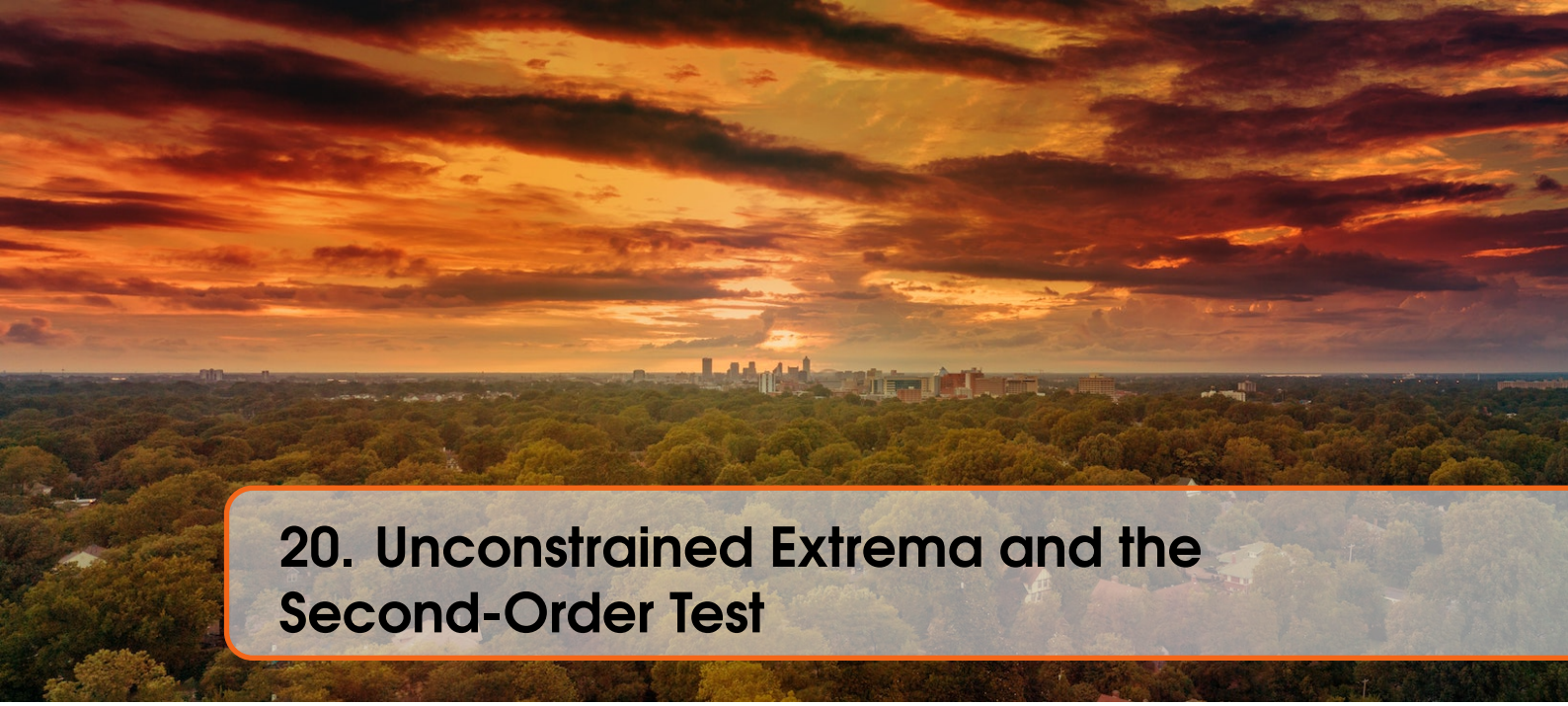
$$y = \varphi(x)$$

such that

$$F(x, \varphi(x)) = 0.$$

The derivative is

$$D\varphi(x) = -[D_y F(x, \varphi(x))]^{-1} D_x F(x, \varphi(x)).$$



20. Unconstrained Extrema and the Second-Order Test

$f : \mathbb{R}^n \to \mathbb{R}$ is differentiable and x_* is an extreme internal place:

$$\nabla f(x_*) = 0.$$

$f \in C^2$ and $\nabla f(x_*) = 0$:

- Hess $f(x_*) \succ 0 \Rightarrow$ minimum local narrow;
- Hess $f(x_*) \prec 0 \Rightarrow$ maximum local tight;
- Hessian indefinite $\Rightarrow$ saddle point;
- Hessian semidefinite $\Rightarrow$ inconclusive tests in general.



21. Table of Main Implications

Assumption	**Consequence**
Fréchet differentiable in x_0	continues x_0
C^1 close to x_0	differentiable in x_0
Lipschitz	uniformly continuous
Df limited on convex set	f Lipschitz
∇f L-Lipschitz	linear rest $O(\|h\|^2)$
Hess f Lipschitz	Taylor order 2 with rest $O(\|h\|^3)$
$f = o(g)$	$f = O(g)$
$f \sim g$	$f = g + o(g)$
$f \in C^2$	Hessian symmetric
$\det Df(x_0) \neq 0$	local invertability



22. Hierarchy of Regularity Notions

In general:

$$C^1 \Longrightarrow \text{differentiable} \Longrightarrow \text{continuous.}$$

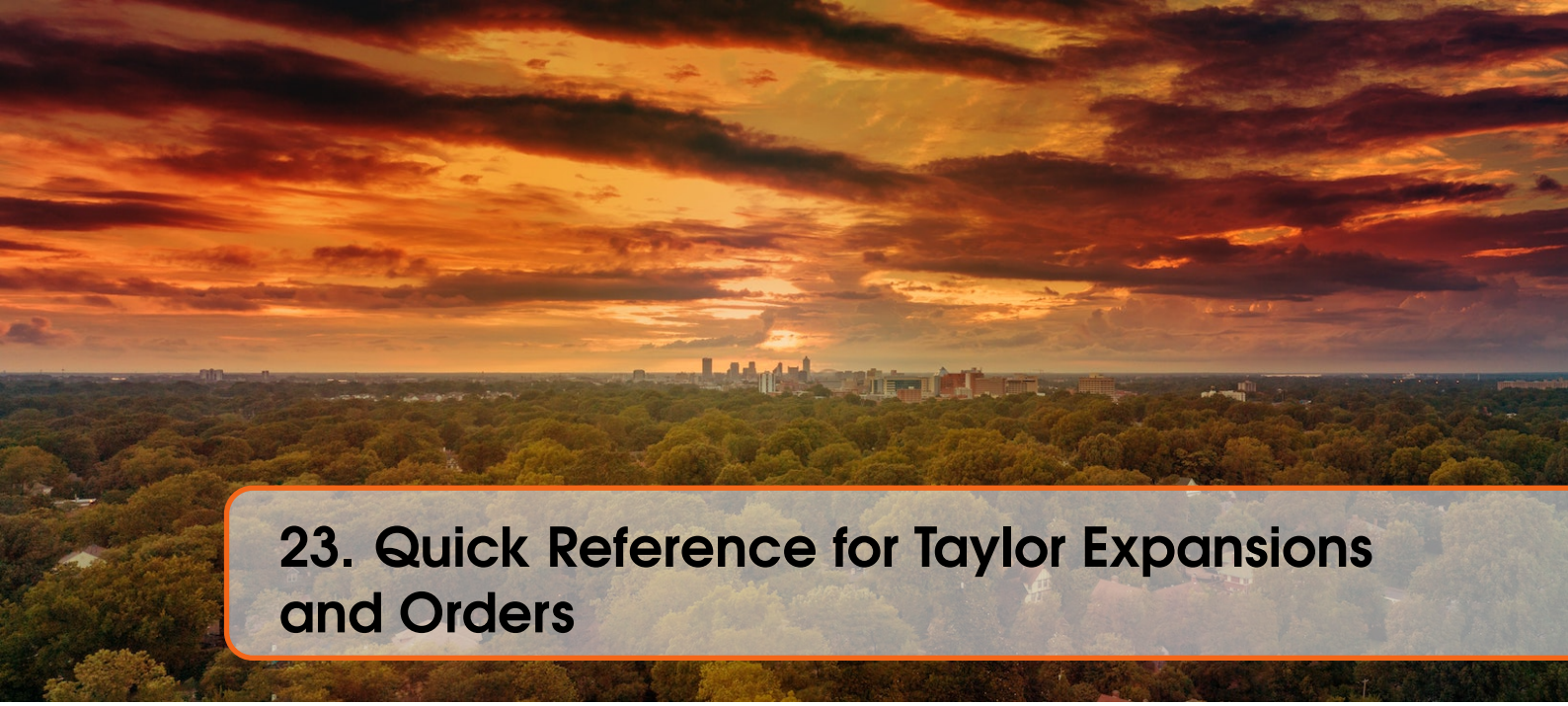
On convex sets, if Df is limited:

$$C^1 + \sup \|Df\| < \infty \Longrightarrow \text{Lipschitz.}$$

For $0 < \alpha < 1$:

$$\text{Lipschitz} \Longrightarrow C^{0,\alpha} \text{ locally on bounded sets.}$$

But, in general, none of the inverse implications is valid without additional hypotheses.



23. Quick Reference for Taylor Expansions and Orders

Regularity	**Local expansion**
Differentiated	$f(x+h) = f(x) + Df(x)h + o(\|\|h\|\|)$
Df locally Lipschitz	$f(x+h) = f(x) + Df(x)h + O(\|\|h\|\|^2)$
Twice differentiable	$f(x+h) =$ $f(x) + Df(x)h + \frac{1}{2}D^2f(x)[h,h] + o(\|\|h\|\|^2)$
D^2f locally Lipschitz	$f(x+h) =$ $f(x) + Df(x)h + \frac{1}{2}D^2f(x)[h,h] + O(\|\|h\|\|^3)$
C^{p+1} locally	$f = T_p + O(\|\|h\|\|^{p+1})$
Taylor-Peano order p	$f = T_p + o(\|\|h\|\|^p)$



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