Ordinary Differential Equations and Systems Dictionary

Theory, Stability, and Boundary-Value Problems

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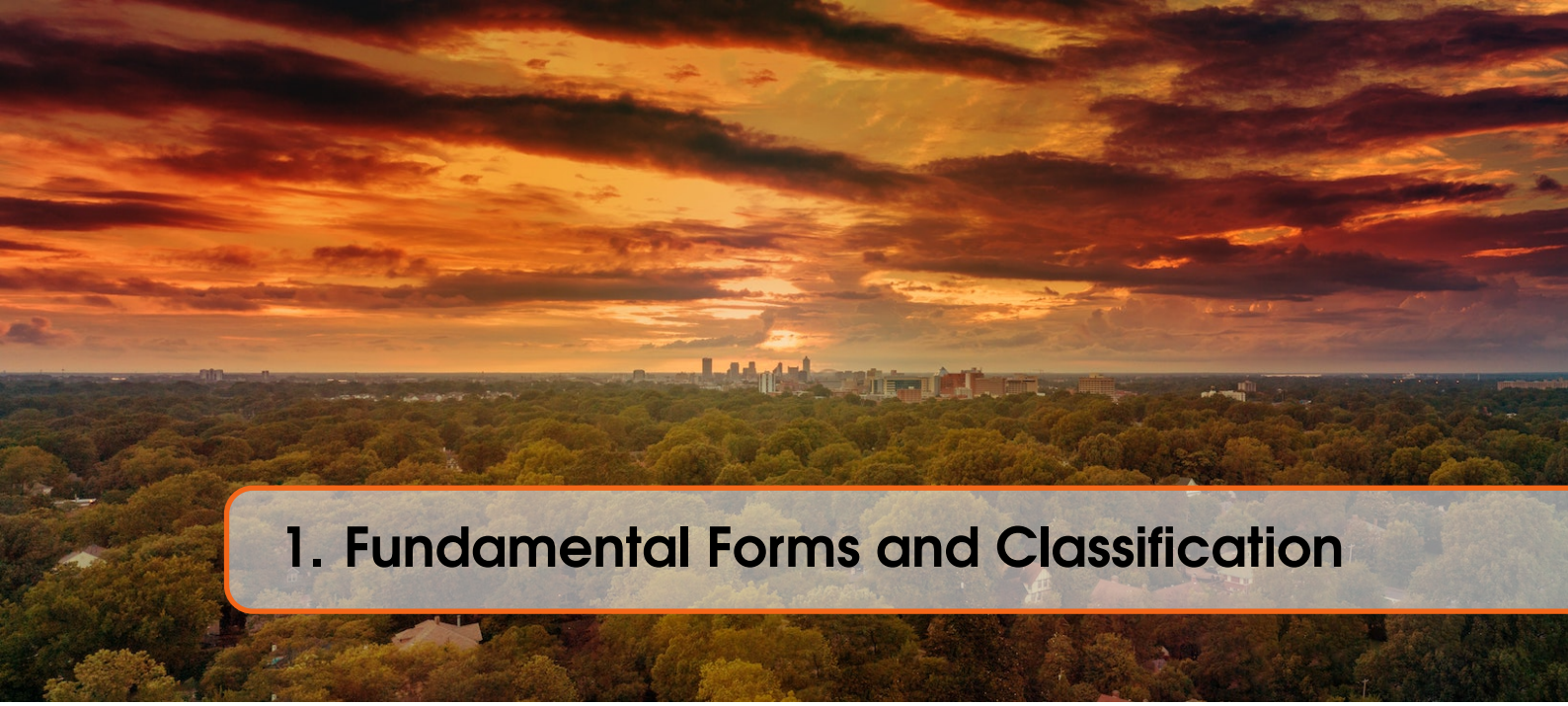
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1. Fundamental Forms and Classification

Purpose and Scope

This is not a set of lecture notes. It is a compact dictionary of definitions and results used in the accompanying courses. If you want to study any of these topics, you should follow a course or consult a textbook devoted to the subject. This dictionary is intended for readers who already know, or have previously studied, the material and need a concise reference whenever a reminder is useful.

An ODE of order p can be written as

$$F(t,y,y',\dots,y^{(p)})=0, \qquad y^{(p)}=f(t,y,\dots,y^{(p-1)}).$$

Setting $x=(y,y',\dots,y^{(p-1)})^T$ gives a first-order system

$$x'=f(t,x), \qquad x\in\mathbb{R}^n.$$

The initial-value problem (IVP), or Cauchy problem, is

$$\boxed{x'=f(t,x), \qquad x(t_0)=x_0.}$$

An autonomous system has the form $x'=f(x)$; a nonautonomous system has the form $x'=f(t,x)$. A linear system is

$$x'=A(t)x+b(t),$$

and is homogeneous when $b=0$ and has constant coefficients when $A(t)=A$.

1.1 Important Scalar ODEs

A separable equation satisfies

$$y'=a(t)b(y), \qquad \int \frac{dy}{b(y)}=\int a(t)\,dt+C.$$

Zeros of b may give constant solutions. For a linear equation,

$$y'+a(t)y=b(t), \qquad \mu(t)=\exp\left(\int a(t)\,dt\right),$$

the integrating-factor formula is

$$y(t) = \mu(t)^{-1}\left[C + \int \mu(t) b(t)\, dt\right].$$

A Bernoulli equation and its standard substitution are

$$y' + a(t)y = b(t)y^{\alpha}, \quad \alpha \neq 0, 1, \qquad z = y^{1-\alpha}.$$

A Riccati equation is

$$y' = a(t) + b(t)y + c(t)y^2.$$

Given a particular solution y_p, the substitution $y = y_p + 1/z$ produces a linear equation. An exact equation satisfies

$$M(t,y)\, dt + N(t,y)\, dy = 0, \qquad M_y = N_t,$$

and locally has the form $\Phi(t,y) = C$, with $\Phi_t = M$ and $\Phi_y = N$.



2. Existence, Uniqueness, and Continuation

A classical solution $x : I \to \mathbb{R}^n$ is continuously differentiable and satisfies the ODE pointwise. Its integral form is

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds.$$

2.1 Peano's Theorem

Continuity of f implies local existence, but not necessarily uniqueness.

2.2 Picard–Lindelöf Theorem

If f is continuous in t and locally Lipschitz in x, the IVP has a unique local solution. The Lipschitz condition is

$$\|f(t,x) - f(t,y)\| \le L\|x - y\|.$$

If f is C^1 with respect to x, it is locally Lipschitz in x. Picard iteration is

$$x^{(k+1)}(t) = x_0 + \int_{t_0}^{t} f(s, x^{(k)}(s))\, ds.$$

2.3 Carathéodory Solutions

Carathéodory theory permits measurable dependence on t and continuous dependence on x, together with suitable bounds. Solutions are absolutely continuous and satisfy the ODE almost everywhere.

2.4 Maximal Interval

The unique solution extends to a maximal interval (α, β). It can be continued as long as its graph remains in a compact subset of the domain of f. At-most-linear growth,

$$\|f(t,x)\| \le a(t) + b(t)\|x\|,$$

provides standard conditions for global existence.



3. Dependence on Data and Grönwall's Inequality

If f is Lipschitz in x, then

$$\|x(t) - y(t)\| \leq e^{L|t-t_0|}\|x(t_0) - y(t_0)\|.$$

Grönwall's inequality states that

$$u(t) \leq a + \int_{t_0}^{t} b(s)u(s)\,ds \quad \Longrightarrow \quad u(t) \leq a\exp\left(\int_{t_0}^{t} b(s)\,ds\right).$$

For $x' = f(t,x,\mu)$, sufficient regularity gives continuous or differentiable dependence of the solution on initial data and parameters.



4. Flows, Variational Equations, and Sensitivities

For $x' = f(x)$, the flow satisfies

$$\varphi_0 = I, \qquad \varphi_{t+s}(x) = \varphi_t(\varphi_s(x)).$$

For nonautonomous systems one uses $\Phi(t,s,x)$ with the corresponding composition property. Linearization along $x(t)$ gives

$$\delta x' = \nabla_x f(t, x(t))\, \delta x.$$

The sensitivity with respect to the initial condition satisfies

$$S = \frac{\partial x(t; t_0, x_0)}{\partial x_0}, \qquad S' = \nabla_x f\, S, \quad S(t_0) = I.$$

Parameter sensitivity satisfies

$$S'_\mu = \nabla_x f\, S_\mu + f_\mu.$$



5. Nonautonomous Linear Systems

For

$$x' = A(t)x + b(t),$$

a fundamental matrix satisfies

$$\Phi' = A(t)\Phi, \qquad \det\Phi \neq 0.$$

The state-transition matrix obeys

$$\Phi(s,s) = I, \qquad \Phi(t,r)\Phi(r,s) = \Phi(t,s), \qquad \Phi(t,s)^{-1} = \Phi(s,t).$$

Variation of constants gives

$$\boxed{x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,s)b(s)\,ds.}$$

The Wronskian is

$$W(t) = \det[x_1(t) \ \cdots \ x_n(t)],$$

and Liouville's formula is

$$\det\Phi(t) = \det\Phi(t_0)\exp\left(\int_{t_0}^{t} \operatorname{tr}A(s)\,ds\right).$$



6. Constant-Coefficient Linear Systems

For $x' = Ax$,

$$x(t) = \mathrm{e}^{A(t-t_0)}x_0.$$

The matrix exponential satisfies

$$\mathrm{e}^{At} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}, \qquad \frac{d}{dt}\mathrm{e}^{At} = A\mathrm{e}^{At}, \qquad (\mathrm{e}^{At})^{-1} = \mathrm{e}^{-At},$$

$$\det(\mathrm{e}^{At}) = \mathrm{e}^{t\,\mathrm{tr}\,A}.$$

If $AB = BA$, then $\mathrm{e}^{A+B} = \mathrm{e}^A\mathrm{e}^B$. If $A = V\Lambda V^{-1}$, then

$$\mathrm{e}^{At} = V\mathrm{e}^{\Lambda t}V^{-1}.$$

For $J = \lambda I + N$ with $N^p = 0$,

$$\mathrm{e}^{Jt} = \mathrm{e}^{\lambda t}\sum_{k=0}^{p-1} \frac{t^k N^k}{k!}.$$

The forced system has the solution

$$x(t) = \mathrm{e}^{A(t-t_0)}x_0 + \int_{t_0}^{t} \mathrm{e}^{A(t-s)}b(s)\,ds.$$

If b is constant and A is invertible,

$$x_\star = -A^{-1}b, \qquad x(t) = x_\star + \mathrm{e}^{A(t-t_0)}(x_0 - x_\star).$$



7. Higher-Order Scalar Linear ODEs

Consider

$$a_p(t)y^{(p)} + \cdots + a_0(t)y = g(t).$$

For constant coefficients, the characteristic polynomial is

$$p(\lambda) = a_p\lambda^p + \cdots + a_0.$$

A real root λ of multiplicity m gives

$$e^{\lambda t}, \quad te^{\lambda t}, \quad \ldots, \quad t^{m-1}e^{\lambda t}.$$

A conjugate pair $\alpha \pm i\beta$ gives

$$e^{\alpha t}\cos\beta t, \qquad e^{\alpha t}\sin\beta t.$$

The superposition principle is

$$L[y] = g, \qquad y = y_h + y_p.$$



8. Autonomous Systems and Phase Space

For $x' = f(x)$, an equilibrium $x_\star$ satisfies

$$f(x_\star) = 0.$$

A set S is positively invariant if $x_0 \in S$ implies $\varphi_t(x_0) \in S$ for every $t \geq 0$. A first integral satisfies

$$\frac{d}{dt} I(x(t)) = \nabla I(x)^T f(x) = 0.$$



9. Stability of Linear Systems

For $x' = Ax$,

$$\text{asymptotically/exponentially stable} \iff \operatorname{Re}\lambda < 0 \quad \text{for every } \lambda \in \operatorname{spec}(A).$$

If some eigenvalue has positive real part, the system is unstable. If all eigenvalues have nonpositive real part, stability requires the eigenvalues on the imaginary axis to be semisimple.



10. Gradient, Conservative, and Hamiltonian Systems

For a gradient flow,

$$x' = -\nabla V(x), \qquad \frac{dV}{dt} = -\|\nabla V(x)\|^2 \le 0.$$

A Hamiltonian system has the form

$$z' = J\nabla H(z), \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

If H is autonomous, then $dH/dt = 0$. The flow is symplectic:

$$(D\varphi_t)^T J (D\varphi_t) = J.$$



11. Boundary-Value Problems

A boundary-value problem (BVP) is

$$x' = f(t,x), \qquad B(x(a),x(b)) = 0.$$

The linear case is

$$x' = A(t)x + b(t), \qquad B_a x(a) + B_b x(b) = c.$$

Shooting parameterizes the missing initial data and solves a residual equation $F(s) = 0$. Multiple shooting uses several IVPs linked by continuity conditions. Collocation enforces the ODE at selected points of a finite-dimensional approximation.



12. Final Concept Map

$$\boxed{x' = f(t,x) \Longrightarrow \{\text{existence, uniqueness, flow, sensitivity, stability, numerics}\}.}$$

$$\boxed{x' = A(t)x + b(t) \Longrightarrow \{\Phi(t,s),\ \text{Liouville, variation of constants}\}.}$$

$$\boxed{x' = Ax \Longrightarrow \{e^{At}, \operatorname{spec}(A),\ \text{Jordan form, linear stability}\}.}$$

$$\boxed{x' = f(x),\ f(x_\star) = 0 \Longrightarrow \{\text{equilibrium, invariants, autonomous dynamics}\}.}$$



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